Moduli, Motives, Mirrors

Yuri I. Manin

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1. Introduction

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1.1. Brief history

Mathematical history of Mirror Symmetry started in 1991, when an identity of a new type was discovered in the ground-breaking paper by four physicists [20] (it was reproduced in [85] where earlier works are also described and motivated).

The left-hand side (or A-side) of this identity was a generating series for the numbers \( n(d) \) of rational curves of various degrees \( d \) lying on a smooth quintic hypersurface in \( \mathbb{P}^4 \). The right-hand side (B-side) was a certain hypergeometric function. The Mirror Identity states that the two functions become identical after an explicit change of variables which is defined as a quotient of two hypergeometric functions of the same type.

At the moment of discovery, not only the identity itself remained unproved, but even its A-side was not well defined: the correct way of counting rational curves was proposed by M. Kontsevich ([60]) only in 1994. In the same remarkable paper Kontsevich gave an explicit formula for \( n(d) \) creatively using Bott’s fixed point formula for torus actions at the target space. After the appearance of this paper one could hope that the Mirror Identity for quintics (and more general toric submanifolds) ought to be provable by algebraic manipulations with both sides. This turned out to be a difficult problem. A. Givental brought this program to a successful completion in 1996, by introducing a new torus action at the source space, stressing equivariant cohomology and inventing ingenious calculational strategy (see [40, 43, 45, 15, 90]). For subsequent important developments, see [66, 67, 14].
This work however did not unveil the mystery of the Mirror Identity. The point is that the identity itself was discovered by the physicists as only one manifestation of a deeper principle. Physicists believe that with any Calabi-Yau manifold $X$ one can associate two $\mathcal{N} = (2,2)$ Superconformal Field Theories (SCFT) which are the respective $A$ and $B$ models (see e.g. [105]). The Mirror Correspondence between $X$ and $Y$ supposedly interchanges their $A$ and $B$ models. In particular, in the case of quintics the hypergeometric functions involved are actually periods of the mirror partner family of our quintics, and $B$-models generally reflect properties of variations of periods and Hodge structures.

Unfortunately, a precise and complete mathematical definition of what constitutes an $\mathcal{N} = (2,2)$ ScFT is still lacking. Various components of this structure with varying degree of precision are described in the papers collected in [85] and [86]. In particular, a part of this structure is a modular functor in the sense of Segal, with possibly infinite dimensional Hilbert space. In turn, such theories are often constructed via representation theory of a vertex algebra. See [73] and [17, 18] for the most recent mathematical approach to this picture, achieving at least the construction of what seems to be the right vertex algebra.

The parts that are involved in the statement of Mirror Identity above refer correspondingly to the Quantum Cohomology ($A$-model, physicists' $\sigma$-model) and extended variations of Hodge structure. Both are now well understood mathematically: see [79] and [4] respectively. However, the Mirror partners are connected by many more ties than a mere Mirror Identity. These ties, in particular, relate Lagrangian and complex geometry in a remarkable way: see [96] and [58] for the basic conjectures to this effect.

Therefore now, more than a decade after it was discovered, the Mirror Symmetry mathematically looks like a complex puzzle, some of the pieces of which have found their respective places, some are still lying in disorder, and some, most probably, are missing.

1.2. Plan of the paper

This puzzle metaphor guided the organization of this report.

Section 2 is devoted to the binary relation of mirror partnership between families of Calabi-Yau manifolds endowed with additional structures which we call here cusps. This relation consists in the isomorphism of two Frobenius manifolds, constructed in two different ways for the respective families. In turn, Frobenius manifold isomorphisms generalize the Mirror Identity of [20].

Section 3 explains various versions of another mirror partnership relation, this time between certain symplectic, on the one hand, and complex, on the other hand, manifolds, endowed with additional structure which in this case is a choice of a fibration by real tori. Here I have taken as starting point a part of Kontsevich's package [58], with further details taken from [96, 93, 2], and other papers. I have chosen for these relations the word "partnership", or "duality", as opposed to "symmetry", because the definitions of both of them are explicitly un-symmetric.