Progress in Mathematics, Vol. 196


# Loop spaces of configuration spaces, braid-like groups, and knots

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**Abstract.** The purpose of this note is to describe some relationships between the following topics: (1) higher dimensional variations of braids, (2) loop space homology, (3) Hopf algebras given by loop space homology, (4) natural groups attached to connected Hopf algebras, (5) analogues of Artin's (pure) braid group, (6) Alexander's construction of knots arising from loop spaces, and (7) Vassiliev's invariants of braids.

## 1. Introduction

The purpose of this article is to give higher dimensional analogues of braids as well as an analogue of "braiding" certain "pieces" of a manifold such as hyperplanes in Euclidean space or projective spaces in certain Lie groups. The main direction here is that one obtains a Hopf algebra via the homology of the loop space of a configuration space. These Hopf algebras then give groups which have properties that are analogous to Artin's (pure) braid group, and arise from classical algebraic topology. This article is a survey of how some of these results fit [7, 9, 11, 12, 13, 24]. A smattering of new results are included.

These Hopf algebras are special cases of a version of "braiding" which occur in a broader context. In addition to the "infinitesimal braid relations" to be defined below, there are additional relations depending on the underlying geometry of the manifold. The Lie algebras obtained in this way depend on certain naive features of the underlying manifold. Some examples are given below. One natural family of manifolds arises from the Lie groups $\mathrm{SU}(n)$. The structures of the Hopf algebras, Lie algebras, and groups encountered here differ drastically in case of $\mathrm{SU}(3)$ than those for $\mathrm{SU}(n)$ with $n$ not 3.

One of the Hopf algebras that occurs here is the universal enveloping algebra of the "universal Yang–Baxter Lie algebra" which satisfies relations that are sometimes called the "infinitesimal braid relations", or the "horizontal 4T relation" and "framing independence" to knot theorists. This Hopf algebra is the universal enveloping algebra of a graded Lie algebra which also occurs in the study of the

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1991 *Mathematics Subject Classification.* Primary: 55P35, Secondary: 14D99.
*Partially supported by the NSF.

J. Aguadé et al. (eds.), *Cohomological Methods in Homotopy Theory*

Vassiliev invariants of braids [3, 18, 19, 20, 26]. In particular, these algebras "account" for all of the Vassiliev invariants of braids as described below in Sections 4, 5 and 9.

The Hopf algebras encountered here give rise to natural groups. Namely, attached to any Hopf algebra with conjugation (or antipode) is the group of coalgebra maps from a fixed coalgebra to the Hopf algebra. With a natural fixed choice of coalgebra given below, this group is filtered, and the underlying set of the associated graded is the product of the underlying set of primitive elements in the Hopf algebra. In the special case for which the Hopf algebra is the homology of the loop space of the configuration space for $\mathbb{R}^{2n}$, the associated graded is a Lie algebra that is isomorphic to the Lie algebra associated to the descending central series for the pure braid group (where these Lie algebras are tensored with the rational numbers). There are further Lie algebras which take into account the underlying geometry of the manifold.

With a particular choice of coalgebra, the groups alluded to above are obtained by assembling the images of the Hurewicz homomorphism into a natural group which, as a set, is the product of the primitive elements in a Hopf algebra. One example of these groups of coalgebra maps is given by the Mal'cev completion of a free group [24, 16]. These completions assemble themselves in various "twisted" ways in the case where the Hopf algebra is the homology of the loop space of the configuration space for $\mathbb{R}^n$.

Namely, the associated Lie algebras are isomorphic (in characteristic zero) to the "universal Yang–Baxter Lie algebra", but the groups themselves have different structures. The point of this is that the Hopf algebras attached to these constructions "see" the same quadratic commutator relations as do the pure braid groups. The precise relations in the group of coalgebra maps also agrees with those of the pure braid group through "quadratic terms". However, the groups of coalgebra maps are filtered and are isomorphic on the level of associated graded modules, but there are terms of higher filtration which appear in the precise relations for groups of coalgebra maps, and the groups themselves are not actually isomorphic to the braid groups [24].

Regarding braids as equivalence classes of motions of distinct particles in the plane through time, there is an extension given by replacing points by other "pieces" of a manifold. There are natural "braid-like" groups which are defined for any manifold which reflect these motions, and arise as a target of the classical Hurewicz homomorphism. These "braid-like" groups satisfy relations given in the braid group up to "quadratic terms" as well as "extended Yang–Baxter relations" as described in Section 3 provided that the underlying manifold enjoys additional naive geometric properties. These structures are developed in Sections 5 and 6.

Constructions involving loop spaces of configuration spaces in turn inform on spaces of embeddings. For example, a classical result of J. W. Alexander gives that every "knot type" is given by "closing-up" a choice of braid [1]. Alexander's construction arises as a map from the loop space of a configuration space to the space of embeddings of $S^1$ in $\mathbb{R}^3$, and fits in a wider context. That is, there is a