Chapter IV

Stationary Schrödinger Processes

We have formulated non-relativistic quantum mechanics as a diffusion theory in the preceding chapters, but ignored to some extent the mathematical problems that arise in connection with the singularity of coefficients of diffusion equations, since we have concentrated on clarifying fundamental concepts and mathematical structures. We shall see that our diffusion processes, Schrödinger diffusion processes, must be treated more carefully. We will analyze some typical stationary cases to see the movement of quantum particles.

4.1. Stationary States

We will treat equations with time-independent potentials in stationary states. We consider the equation of motion

\[
\frac{\partial \phi}{\partial t} + \frac{1}{2} \sigma^2 \Delta \phi + c(x) \phi = 0, \quad (4.1.1)
\]

\[- \frac{\partial \hat{\phi}}{\partial t} + \frac{1}{2} \sigma^2 \Delta \hat{\phi} + c(x) \hat{\phi} = 0,
\]

where \(c(x) = -V(x)\) by equation (3.7.5) with a given potential function \(V(x)\), or equivalently the Schrödinger equation

\[
i \frac{\partial \psi}{\partial t} + \frac{1}{2} \sigma^2 \Delta \psi - V(x) \psi = 0, \quad (4.1.2)
\]

\[-i \frac{\partial \bar{\psi}}{\partial t} + \frac{1}{2} \sigma^2 \Delta \bar{\psi} - V(x) \bar{\psi} = 0.
\]

Through substitution of

\[
\phi = e^{i\lambda t} \varphi(x), \quad \text{and} \quad \psi = e^{-i\lambda t} \varphi(x),
\]
respectively, the equation of motion and the Schrödinger equation are reduced to the same eigenvalue problem

\[
-\frac{1}{2} \sigma^2 \Delta \varphi + V(x) \varphi = \lambda \varphi. \tag{4.1.3}
\]

The distribution of the Schrödinger diffusion process \( \{X_t, Q\} \) is then given by

\[
Q[X_t \in dx] = \phi_t(x) \varphi_t(x) dx = \psi_t(x) \psi_t(x) dx = |\varphi(x)|^2 dx, \tag{4.1.4}
\]

in terms of solutions \( \varphi(x) \) of equation (4.1.3), which is independent of time.

### 4.2. One-Dimensional Harmonic Oscillator

We consider a simple case of the one-dimensional harmonic oscillator, taking a potential function

\[
V(x) = \frac{1}{2} \kappa^2 x^2. \tag{4.2.1}
\]

Then the eigenvalue problem in (4.1.3) turns out to be

\[
-\frac{1}{2} \sigma^2 \frac{d^2 \varphi(x)}{dx^2} + \frac{1}{2} \kappa^2 x^2 \varphi(x) = \lambda \varphi(x), \tag{4.2.2}
\]

where we require the boundary condition \( \lim_{x \to \pm \infty} \varphi(x) = 0 \). It is well-known that solutions must be of the form\(^1\)

\[
H_n(\sqrt{\kappa/\sigma} x) e^{-\kappa x^2/(2\sigma)},
\]

where

\[
H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.
\]

Some of them are

\[
H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 2(2x^2 - 1),
\]

\[
H_3(x) = 4(2x^3 - 3x), \quad H_4(x) = 4(4x^4 - 12x^2 + 3),
\]

\[
H_5(x) = 8(4x^5 - 20x^3 + 15x).
\]

\(^1\) Cf. e.g. chapter III of Pauling-Wilson (1935).