CHAPTER VI

CANONICAL OPERATORS ON LAGRANGIAN MANIFOLDS
WITH COMPLEX GERM AND THEIR APPLICATIONS TO
SPECTRAL PROBLEMS OF QUANTUM MECHANICS

In this chapter we shall consider some specific applications of complex
germs theory to spectral problems in quantum mechanics for scalar Schrödinger
and Klein-Gordon operators. In spectral problems with small parameter \( h \), the
spectrum is very dense, and, as a rule, there are no general asymptotics for all
eigenvalues \( E_n \) as \( n \to \infty \); this is a consequence of the fact that this problem has
several parameters: \( h \) and the number of eigenvalues \( n \), which can be a multi­
index \( n = (n_1, n_2, n_3) \). The form of the asymptotics for \( E_n = E_n(h) \) depends
on the relations between \( h \) and \( n \), i.e., the spectrum of a spectral problem
with a small parameter \( h \) is divided into series. In some problems these series
can be classified. In particular, such a classification can be constructed for
the operators of quantum mechanics by using the trajectories of Hamiltonian
systems.

Using our general method (see the previous chapters), we shall construct
the semiclassical spectral series corresponding in the limit as \( h \to 0 \) to the
motion of a charged particle along closed trajectories, which are the projections
on the configuration space of one-dimensional Lagrangian tori (the latter are
closed phase trajectories of the classical Hamiltonian system corresponding to
the quantum one).

In an arbitrary electromagnetic field, the classical Hamiltonian system
cannot be integrated, and the problem of constructing a family of closed tra­
jectories is a nontrivial problem of intrinsic interest. Moreover, even in the case
of an integrable system, the requirement that the phase trajectories must be
closed is a very rigid condition. For example, it is well-known that in the case
of motion in the field of central forces, \( H = p^2/2 + V(|x|) \), \( p \in \mathbb{R}^3_p \), \( x \in \mathbb{R}^3_x \), all
bounded trajectories are closed only for either \( V(|x|) = |x|^2 \) or \( V(|x|) = |x|^{-1} \)
(see [3, 4]).

Nevertheless, rather often specific physical systems possess additional
symmetries. They generate first integrals of the Hamiltonian system which,
possibly, are not even in involution with respect to the Poisson brackets. At

\(^1\)The results of this chapter were obtained jointly with my pupils S. Yu. Dobrokhotov
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present such systems are intensively studied [4, 84, 101, 110]. The additional symmetries allow to reduce the initial Hamiltonian system to other simpler problems, for example, to the problem of finding the stationary points of a Hamiltonian on a reduced phase space, instead of the problem of finding the closed trajectories of the initial Hamiltonian [4, 84].

Here we shall restrict ourselves to the case of systems where, as a rule, there is only one cyclic (angular) coordinate related to the configuration space. A typical example is provided by electromagnetic fields that admit the group of rotations generating the first integral (the projection of the kinetic angular momentum on an axis chosen in the configuration space).

In a Hamiltonian system with a cyclic (angular) variable, one can always single out a special family of closed trajectories, namely, circles representing stationary motion or points of relative equilibrium of the initial system [4, 110]. This type of motion appears in many applied problems. Thus it is expedient to consider it separately (see §1 of this chapter), especially because for such systems of trajectories one can investigate the existence and uniqueness problems for invariant complex germ in a sufficiently constructive way. The latter is important for determining the subsequent terms of the asymptotic expansions. Nonuniqueness of the complex germ results in the degeneration of the spectrum for the problem in the zero approximation and requires a separate investigation of the right choice of functions in the zero approximation in each concrete case.

In some specific situations the problems of finding the spectral series (semiclassical eigenfunctions and eigenvalues) for quantum Hamiltonians corresponding to closed trajectories are of independent physical interest [21, 107, 108, 111]. In some particular cases such series were obtained by separating variables in the harmonic approximation. For example, the spectral series of the Klein-Gordon and Dirac operators were constructed in this way. These series, corresponding to the motion of relativistic electrons along a stable circle in a cyclic accelerator with weak focusing, play an important role in the quantum theory of synchrotron radiation [22, 107, 108, 111, 113]. Similar series in axial fields are used for calculating certain effects in quantum electrodynamics, for example, the effect of radiational self-polarization of electron-positron beams in crystals in the axial channelling model [1, 15, 22, 32]. With the help of the complex germ theory, we can construct similar spectral series in these fields as well as in some other important model electromagnetic fields (see the table below).

In the previous chapters we developed the theory of quantization for stationary points and closed curves, namely, for zero-dimensional and one-dimensional Lagrangian manifolds (tori). At the same time, the Hamiltonian systems possess invariant tori of dimension \( k \), \( k \geq 2 \). The maximal dimension of Lagrangian manifolds is equal to the dimension of the configuration space of the system. In this case they are called complete-dimensional [88]. Complete-dimensional Lagrangian tori can be quantized by the methods developed in [88, 89] (see also [34, 44]). Some examples of quantization of complete-dimensional