The discrete spectrum in a gap of the continuous one for compact supported perturbations

M.Sh. Birman T. Weidl

1. We start from the traditional problem on the negative spectrum of the Schrödinger operator in $\mathbb{R}^d$, $d \geq 3$. Let $A = -\Delta$,

$$A(\alpha) = -\Delta - \alpha V, \quad V(x) \geq 0, \quad \alpha > 0,$$

(1)

and $\lambda \leq 0$. By $N_A(\alpha, \lambda)$ we denote the number of eigenvalues of the operator (1) on the left-hand side of the point $\lambda$. Then for potentials $V \in L_{d/2}(\mathbb{R}^d)$ we have the well-known asymptotics

$$N_A(\alpha, \lambda) \sim (2\pi)^{-d} \omega_d \alpha^{d/2} \int V^{d/2} dx, \quad \alpha \to \infty,$$

(2)

with $\omega_d$ the volume of the unit ball in $\mathbb{R}^d$. We call potentials $V \in L_{d/2}(\mathbb{R}^d)$ "regular" perturbations of the operator $A$ (cf. [BS1]). The asymptotic (2) do not depend on $\lambda \leq 0$, it's character is determined by the behavior of the symbol $|\xi|^2 - \alpha V(x)$ for large $|\xi|$ only. In [BS1] the asymptotics of $N_A(\alpha, \lambda)$ are discussed precisely for potentials violating the assumption $V \in L_{d/2}(\mathbb{R}^d)$ because of a slow decrease as $|x| \to \infty$ ("non-regular perturbations"). There typically $N_A(\alpha, \lambda) = o(N(\alpha, 0)), \lambda < 0$, is found; the main asymptotical term of $N(\alpha, 0)$ for $\alpha \to \infty$ is given by the symbol of $A(\alpha)$ for small $|\xi|$ (threshold effect). So, for instance, for $V \in L_\infty, V(x) \sim |x|^{-2}(\ln |x|)^{-1/q}, \quad 2q > d, \quad |x| \to \infty$, we have $N(\alpha, 0) \sim c(d)\alpha^q, \quad N(\alpha, \lambda) = O(\alpha^{d/2} \ln \alpha)$, and the latter estimate can be refined. Here we discuss the inverse case, when $V \notin L_{d/2}(\mathbb{R}^d)$ because of local singularities. In detail we assume

$$V \in L_1(\mathbb{R}^d), \quad \text{supp} \ V \subset K_R := \{x : |x| < R\}, \quad V \geq 0.$$

(3)

We call potentials of the form (3) "quasi-regular".

2. For $V \notin L_{d/2}(\mathbb{R}^d)$ the number of eigenvalues $N_A(\alpha, \lambda)$ can show non-powerlike asymptotics. Our second aim is to show that the technical tools developed in [W1], [W2] allow us to consider non-powerlike asymptotics, too. We call a function $f : \mathbb{N} \to \mathbb{R}_+$ a normal estimation function (NEF), if $f \uparrow \infty$ for $n \to \infty$ and if the function $f^\kappa$ is subadditive for some $\kappa > 0$. We introduce the functionals

$$\Delta_f(\lambda, A) := \lim_{\alpha \to \infty} \sup_{\alpha < A} \alpha^{-1} f(N_A(\alpha; \lambda)), \quad (4)$$

$$\delta_f(\lambda, A) := \lim_{\alpha \to \infty} \inf_{\alpha < A} \alpha^{-1} f(N_A(\alpha; \lambda)). \quad (5)$$
Theorem 1. Let (3) be fullfilled and assume, that for some $\lambda \leq 0$ and for a NEF $f$ $\Delta_f(\lambda, A) < \infty$ holds. Then for every $\mu \leq 0$ we have the equalities

$$\Delta_f(\lambda, A) = \Delta_f(\mu, A), \quad \delta_f(\lambda, A) = \delta_f(\mu, A).$$

(6)

In particular (6) is fullfilled for $\lambda < 0, \mu = 0$; this explains why we call potentials $V$ satisfying assumption (3) "quasi-regular". We remark, that under assumptions of theorem 1 the functionals (4), (5) are determined by the behavior of the symbol of $A(\alpha)$ for large $|\xi|$ only.

3. Further we consider the operator

$$H = -\Delta + p(x), \quad p \in L_\infty(\mathbb{R}^d), \quad d \geq 3,$$

(7)
as unperturbed. The spectrum $\sigma(H)$ may be interrupted by gaps. Let $\Lambda = (\lambda_-, \lambda_+)$ be such a gap. For a large class of potentials $V$ decreasing to zero sufficiently fast for $|x| \to \infty$ the spectrum of the perturbed operator

$$H(\alpha) = H - \alpha V, \quad \alpha > 0, \quad V(x) \geq 0,$$
in the gap $\Lambda$ is discrete. For $\lambda, \lambda_- \leq \lambda \leq \lambda_+$, we introduce $N_H(\alpha, \lambda)$ - the number of eigenvalues of $H(t)$ which passed the point $\lambda$ for coupling constant $t$ increasing from 0 to $\alpha$, (for operator $A(\alpha)$ and $\lambda \leq 0$ the function $N_A(\alpha, \lambda)$ coincides with the function $N_A$ from subsection 1). In [B1] an abstract theorem was presented, which gives the equality of the asymptotical functionals $\Delta_f, \delta_f$ for $A, \mu < 0$ and $H, \lambda \in \Lambda$, in the case of powerlike estimation functions $f$. We state here an analogue of this theorem for arbitrary NEF and apply it to the Schrödinger operator. Next we prepare some material required in the corresponding formulations.

4. Let $\mathcal{H}$ be a Hilbert space, $T \in S_\infty(\mathcal{H})$ (i.e. $T$ is a compact operator on $\mathcal{H}$); and let $\{s_k(T)\}_{k \in \mathbb{N}}$ denote the sequence of singular numbers of the operator $T$. For some NEF $f$ we introduce the operator classes

$$\Sigma_f = \{T \in S_\infty : |T|_f := \sup_{n \in \mathbb{N}} s_n(T)f(n) < \infty\}.$$n

The class $\Sigma_f$ is a complete, non-separable space with respect to the quasi-norm $|\cdot|_f$. We denote by $\Sigma^0_f$ the subspace of operators $T \in \Sigma_f$, for which $s_n(T)f(n) \to 0$. The set of finite rank operators is dense in $\Sigma^0_f$. For $T \in \Sigma_f$ we define the functionals

$$\Delta_f(T) := \lim_{n \to \infty} \sup_{n \in \mathbb{N}} s_n(T)f(n), \quad \delta_f(T) := \lim_{n \to \infty} \inf_{n \in \mathbb{N}} s_n(T)f(n).$$n

For $T = T^*$ analogous functionals $\Delta_f^{(\pm)}, \delta_f^{(\pm)}$ can be introduced by the sequences $\{\lambda_n^{(\pm)}(T)\}$, e.g. the sequences of positive eigenvalues of the operator $\pm T$. All six functionals $\Delta_f, \Delta_f^{(\pm)}, \delta_f, \delta_f^{(\pm)}$ are continuous on $\Sigma_f$. In fact they are well defined and continuous on the factor space $\Sigma_f/\Sigma^0_f$, too. The class $\Sigma_f$ is a two-sided ideal in the space of bounded operators on $\mathcal{H}$. The material of this subsection was developed in [W1]. For similar powerlike ideals see [BS2].