Singular perturbations, regularization 
and extension theory

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For nonpositive singular potentials in quantum mechanics it can happen that 
the Schrödinger operator is not essentially self-adjoint on a natural domain of 
definition or not semibounded from below. In this case we have a lot of self-
adjoint extensions each of them is a candidate for the right physical Hamiltonian 
of the system. Hence the problem arises to single out the right physical self-adjoint 
extension. Usually this problem is solved as follows. At first one has to approximate 
the singular potential by a sequence of bounded potentials (cut-off approximation). 
After that one has to show that the arising sequence of Schrödinger operators 
converges in the strong resolvent sense to one of the self-adjoint extensions if 
the cut-off approximation tends to the singular potential. The so determined self-
adjoint extensions is regarded as the right physical Hamiltonian. Very often the 
right physical Hamiltonian coincides with the Friedrichs extension.

With respect to the Schrödinger operator in $L^2(\mathbb{R}^2)$ this problem was dis-
cussed by [3], [4], [5], [9] and [10]. An operator-theoretical investigation of this 
problem was started by Nenciu in [8] and continued by the authors in [7]. In 
the following we continue those abstract investigations. We assume that a semi-
bounded symmetric operator admits a monotonously decreasing sequence of semi-
bounded symmetric operators such that the corresponding sequence of Friedrichs 
extensions converges in the strong resolvent sense to the Friedrichs extension of 
the symmetric operator with which we have started. The problem will be to find 
necessary and sufficient conditions that any other sequence of semibounded self-
adjoint extensions of the decreasing sequence of symmetric operators converges to 
this Friedrichs extension too. Unfortunately, we are unable to solve the problem 
in full generality. This means we have found a necessary condition which must 
be satisfied in order to have the desired convergence. However, we can prove the 
converse only for special sequences of self-adjoint extensions but not for all.

In more detail the problem can be described as follows. Let $A$ and $V$ be two 
nonnegative self-adjoint operators on the separable Hilbert space $\mathcal{H}$. Further, let 
$\mathcal{D} \subseteq \text{dom}(A) \cap \text{dom}(V)$ a dense subset of $\mathcal{H}$ such that

\[
(Vf, f) \leq a(Af, f) + b\|f\|^2, \quad f \in \mathcal{D}, \quad 0 < a, b.
\] (1)
We introduce the abstract operator $\hat{H}_\alpha$

$$\hat{H}_\alpha f = Af - \alpha Vf, \quad f \in \text{dom}(\hat{H}_\alpha) = \mathcal{D}, \quad \alpha > 0. \quad (2)$$

If the coupling constant $\alpha$, $\alpha > 0$, obeys $\alpha < 1/a$, then the operator $\hat{H}_\alpha$ is symmetric, closable and semibounded with lower bound $-\alpha b$. However, the operator $\hat{H}_\alpha$ is in general not essentially self-adjoint.

**Example 1** Let $\mathcal{H} = L^2(\mathbb{R}^1)$ and let $A$ be the usual Laplace operator on $L^2(\mathbb{R}^1)$, i.e. $A = -d^2/dx^2$. By $V$ we denote the multiplication operator arising from the real potential $V(x)$,

$$V(x) = \frac{1}{4\kappa} \frac{1}{|x|^\beta}, \quad 1 \leq \beta \leq 2, \quad \kappa > 0. \quad (3)$$

Let $\mathcal{D} = C_c^\infty(\mathbb{R}^1 \setminus \{0\})$. If $1 \leq \beta < 2$, then for every $\kappa > 0$ there are real numbers $a < 1$ and $b \geq 0$ such that

$$\int_{-\infty}^{\infty} \frac{1}{4\kappa} \frac{1}{|x|^\beta} |f(x)|^2 dx \leq a \int_{-\infty}^{\infty} |f'(x)|^2 dx + b \int_{-\infty}^{\infty} |f(x)|^2 dx. \quad (4)$$

for $\kappa > 0$. If $\beta = 2$, then this is only true for $\kappa > 1$.

**Example 2** Let $\mathcal{H} = L^2(\mathbb{R}^2)$ and let $A$ be the usual Laplace operator on $L^2(\mathbb{R}^2)$, i.e. $A = -\Delta$. Further, let $\Gamma$ be a smooth curve in $\mathbb{R}^2$ which is parameterized by

$$\Gamma = \{(x, y) \in \mathbb{R}^2 : x = \rho(\varphi) \cos \varphi, y = \rho(\varphi) \sin \varphi, 0 \leq \varphi < 2\pi\} \quad (5)$$

where $\rho(\varphi) > 0$ is a smooth function. Again $V$ is the multiplication operator arising from

$$V(x) = \frac{1}{5\kappa} \frac{1}{||x| - \rho(\varphi)||^\beta}, \quad 1 \leq \beta \leq 2, \quad |x| = \sqrt{x^2 + y^2}. \quad (6)$$

We set $\mathcal{D} = C_c^\infty(\mathbb{R}^2 \setminus \Gamma)$. If $1 \leq \beta < 2$, then for every $\kappa > 0$ there are real numbers $a < 1$ and $b \geq 0$ such that

$$\int_{\mathbb{R}^2} \frac{1}{5\kappa} \frac{1}{||x| - \rho(\varphi)||^\beta} |f(x)|^2 dx \leq a \int_{\mathbb{R}^2} |\nabla f(x)|^2 dx + b \int_{\mathbb{R}^2} |f(x)|^2 dx. \quad (7)$$

For $\beta = 2$ this is true only for $\kappa > 1$.

Let us assume that the $\hat{H}_\alpha$ is not essentially self-adjoint. Since $\hat{H}_\alpha$ is semibounded the Friedrichs extension $\hat{H}_\alpha$ exists. Moreover, denoting by $\hat{A}$ the Friedrichs extension of $\hat{A} = A|\mathcal{D}$ it is not hard to see that $\hat{H}_\alpha$ coincides with the form sum of $\hat{A}$ and $-\alpha V$, i.e.

$$\hat{H}_\alpha = \hat{A} + (-\alpha V). \quad (8)$$

In the above examples the Friedrichs extension corresponds to the Dirichlet boundary condition at $x = 0$ for the first example and on $\Gamma$ for the second one.

Next let us introduce a regularizing sequence for the singular perturbation.