Endotrivial modules
and the Auslander-Reiten quiver

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Introduction

As Dade stated it in [8]:
"There are just too many modules over p-groups!"

More precisely, if $P$ is a $p$-group and $R$ a suitable commutative valuation ring, then almost always the group algebra $RP$ is of wild representation type and there is no classification of all its indecomposable modules. Searching for a useful family of modules that could still be classified Dade was led to study endopermutation $RP$-modules, i.e. $RP$-lattices whose $R$-endomorphisms form a permutation $RP$-module. These modules play an important rôle for example in the study of sources of simple modules. The isomorphism classes of indecomposable endopermutation $RP$-modules with vertex $P$ form an abelian group under a multiplication induced by tensor product. For abelian $P$, Dade determined the structure of this group [8]; for non-abelian $P$ Puig [11] proved at least that this group is finitely generated.

On the way to his result Dade classified the indecomposable endotrivial $FP$-modules where $F$ is a field of characteristic $p$. These modules satisfy the stronger property that their $F$-endomorphism ring is isomorphic to the direct sum of the trivial module $F$ and a projective module. It turns out that for abelian $P$ only the Heller modules $\Omega^n(F)$, for $n \in \mathbb{Z}$, are endotrivial. For a while, endotrivial modules which are not of this form were only known for the generalised quaternion, the dihedral and the semidihedral 2-groups; so it was conjectured that only for these 2-groups there were such exceptions [8]. But then, Okuyama found endotrivial modules which were not Heller translates of the trivial module also for other 2-groups and for the extraspecial groups of order $p^3$ and exponent $p$, for odd primes $p$ [10].

In this article we want to provide some explanation for the exceptions in the dihedral and semidihedral case which includes information about the classification in these cases by locating the endotrivial modules in their components of the Auslander-Reiten quiver for $FP$.

We start by proving some results on the Auslander-Reiten component of an endotrivial module and its position in it. Then we deduce the classification of the endotrivial modules for the dihedral 2-groups from the known classification of all the indecomposable modules; they are just the non-projective modules in the components of $F$ and $\Omega(F)$. At the end we make some remarks on the relation to the earlier conjecture.

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For the following we want to fix some notation. By $G$ we will always denote a finite group, and by $F$ a field of characteristic $p$, where $p$ is a prime dividing the order of $G$. Furthermore, all $FG$-modules are finitely generated left modules. For further standard terminology we refer the reader to [3].

1. Preliminaries on endotrivial modules

An $FG$-module $V$ is called an endotrivial module if $V \otimes V^* \simeq F \oplus Q$, where $F$ denotes the trivial $FG$-module and $Q$ is a projective $FG$-module. So obviously, for any endotrivial module $V$ also the modules $\Omega^n(V)$ are endotrivial for all $n \in \mathbb{Z}$. In particular, for any group $G$ all the modules $\Omega^n(F)$ are examples of endotrivial modules.

As we will often work modulo projectives, we will abbreviate the projective-free part of a module $M$ by $\text{core}(M)$. So $\text{core}(M)$ is the module defined (up to isomorphism) by $M \simeq \text{core}(M) \oplus Q$, where $Q$ is projective and $\text{core}(M)$ has no projective summand. Moreover, if $M$ and $N$ are $FG$-modules, we will write $M \equiv N \pmod{\text{projectives}}$ if $\text{core}(M) \simeq \text{core}(N)$.

First we collect some known results on endotrivial modules. We start with a consequence of a theorem of Alperin and Evens [1] which implies that projectivity can be tested on elementary abelian subgroups (see [3]).

Proposition 1.1. Let $V$ be an $FG$-module. Then $V$ is endotrivial if and only if $V_E$ is endotrivial for any elementary abelian $p$-subgroup $E$ of $G$.

It is easy to see that tensoring with an endotrivial module preserves the number of non-projective indecomposable summands. In fact, this property characterises endotrivial modules (see [6]):

Proposition 1.2. Let $V$ be a non-projective $FG$-module. Then the module $V$ is endotrivial if and only if $\text{core}(V \otimes W)$ is indecomposable for all indecomposable $FG$-modules $W$.

The endotrivial modules form an abelian group with respect to tensor products, modulo projectives. This group is finitely generated; more generally, such a result holds even for endopermutation modules (see Puig [11]).

Despite of this result, there is no classification of the indecomposable endotrivial modules for general $p$-groups. The best theorem in this direction was obtained by Dade [8]:

Theorem 1.3. Let $G$ be an abelian $p$-group. Then the indecomposable endotrivial $FG$-modules are exactly the modules $\Omega^n(F)$, where $n \in \mathbb{Z}$.

2. Components of endotrivial modules in the Auslander-Reiten quiver

For the definition of the Auslander-Reiten sequence of an $FG$-module and the Auslander-Reiten quiver for the group algebra $FG$ we refer the reader to [3].

First we want to prove that tensoring an Auslander-Reiten sequence with an endotrivial module gives again an Auslander-Reiten sequence.