Chapter 3

Construction of Finite Matrix Groups

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Abstract

We describe various methods of construction of matrix representations of finite groups. The applications are mainly, but not exclusively, to quasisimple or almost simple groups. Some of the techniques can also be generalized to permutation representations.

3.1 Introduction

It is one thing to determine the characters of a group, but quite another to construct the associated representations. For example, it is an elementary exercise to obtain the character table of the alternating group $A_5$ by first determining the conjugacy classes, then writing down the trivial character and the permutation characters on points and unordered pairs, and using row orthogonality to obtain the irreducibles of degree 4 and 5, and finally using column orthogonality to complete the table. The result (see Table 3.1) shows that there are two characters of degree 3, but how do we construct the corresponding 3-dimensional representations?

In general, we need some more information than just the characters, such as a presentation in terms of generators and relations, or some knowledge of subgroup structure, such as a generating amalgam, or something similar.

Table 3.1: The character table of $A_5$

<table>
<thead>
<tr>
<th>Class name</th>
<th>$1A$</th>
<th>$2A$</th>
<th>$3A$</th>
<th>$5A$</th>
<th>$5B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class size</td>
<td>1</td>
<td>15</td>
<td>20</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>$\tau$</td>
<td>$\sigma$</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>$\sigma$</td>
<td>$\tau$</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>5</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

$\tau = \frac{1}{2}(1 + \sqrt{5}), \sigma = \frac{1}{2}(1 - \sqrt{5})$
If we have a presentation for our group, then in a sense it is already determined, and there are various algorithms which in principle at least will construct more or less any desired representation. The most important and well-known is Todd–Coxeter coset enumeration, which converts a presentation into a permutation representation, and is well described in many places, such as [11]. From a (faithful) permutation representation it is then (in principle) possible to obtain at least any irreducible representation over any finite field, by using ‘Meataxe’ techniques described by Richard Parker [16]. A generalization of these methods to characteristic zero has recently been described by Parker [17]. Basically these methods enable one to chop any given representation into its irreducible constituents, and then tensor representations together to produce new ones to chop up, and so on.

What we are concerned with here, however, is something more basic, namely how to construct a matrix representation of a group from scratch, when no representation at all is known to begin with. We also assume that no presentation is known, or at least that no presentation can be used to produce a sufficiently small permutation representation.

Note: the original title of my lectures was something like “Computer construction of matrix representations of sporadic simple groups over finite fields”, but it gradually became clear that almost every word of the title was redundant, and that “Construction of groups” was the most the various ideas had in common (and even the word “groups” was a trifle restrictive). The present title is a (somewhat unhappy) compromise between the two extremes.

### 3.2 A Small Example

We illustrate the basic ideas by considering the 3-dimensional representations of $A_5$ mentioned above. Now we know that $A_5$ has a subgroup $A_4$ obtained by fixing one of the five points, and that this subgroup is maximal since it has prime index. Fixing another point we obtain a subgroup $A_3$, and by 2-transitivity there is an element of $A_5$ interchanging these two points, and therefore normalizing $A_3$ to $S_3$. By maximality, these subgroups generate $A_5$, and the situation is as shown in Figure 3.1. In the figure, we also give generating permutations for these subgroups, although the figure can equally well be understood at the level of abstract groups.

If we now wish to construct one of the 3-dimensional representations of $A_5$, we can start by constructing its restriction to $A_4$. For simplicity we assume that the characteristic of the underlying field is not 2 or 3. Then it is easy to see that the character restricts to $A_4$ as the unique 3-dimensional irreducible, and the