14 On the martingales which vanish on the set of Brownian zeroes

In the study of the Hilbert transform of Brownian local times, made in Chapter 10, we encountered some Brownian martingales which vanish (≡ take the value 0) on the set $H \equiv \{(t, \omega) : B_t(\omega) = 0\}$ of the zeroes of a one-dimensional BM $(B_t, t \geq 0)$. Such martingales were also encountered naturally in relation with the balayage formula (see, e.g., Azéma-Yor [88]). It thus seems interesting to try and describe the class

$$\mathcal{M}^0 \equiv \{(M_t) \text{ Brownian martingales}; M_t(\omega) = 0 \text{ if } (t, \omega) \in H\}$$

as well as

$$\mathcal{M}^{0, \text{strict}} \equiv \{(M_t) \text{ Brownian martingales}; M_t(\omega) = 0 \text{ iff } (t, \omega) \in H\}.$$ 

More generally, if $T$ denotes a stopping time with respect to the Brownian filtration $(\mathcal{F}_t)$, we shall be interested in the description of $\mathcal{M}_T^0$ and $\mathcal{M}_T^{0, \text{strict}}$, which are defined as above, but with $H$ being now replaced by $H_T = H \cap [0, T]$.

In the course of this study, which was started jointly with J. Azéma [88], and later continued with T. Jeulin, F. Knight, G. Mokobodzki ([89], [90]), we obtained (see Theorem 14.2 below) the double equivalence:

$$\{(X_t) \in \mathcal{M}_T^0\} \iff \{X_\gamma = 0\} \iff E[X_1 \mid \mathcal{F}_\gamma] = 0,$$

where $\gamma = \sup\{t < 1 : B_t = 0\}$, and $\mathcal{F}_\gamma = \sigma\{z_\gamma; z \text{ any } (\mathcal{F}_t) \text{ predictable process}\}$.

This, in turn, gave a motivation for the study of the equation

$$X_\gamma = E[X_{\infty} \mid \mathcal{F}_\gamma]$$

(\ast)

where, now, the unknown process $(X_t)$ is a general uniformly integrable $(\mathcal{F}_t)$ martingale.
This equation may be solved completely, with the help of some martingale representation results in the enlarged filtration \( (\tilde{F}_t) \), which makes \( \gamma \) a stopping time; this is done in Paragraph 14.3.

The phrasing of our resolution of (*) then makes it necessary to study closely the nature of the filtration \( (\tilde{F}_t) \): in particular, we discuss which information must be added to the \( (\tilde{F}_t) \) (Brownian) martingale part \( (\tilde{B}_t) \) of the original Brownian motion \( (B_t) \) in order to recover \( (\tilde{F}_t) \).

A large part of the results found in this chapter admit quite general extensions (see, in particular, [89], Théorème 1) when \( \gamma \) is replaced by \( L \), the end of an \( (\mathcal{F}_t) \) optional set and, say, \( L \) avoids \( (\mathcal{F}_t) \) stopping times, i.e.: \( P(L = T) = 0 \), for any \( (\mathcal{F}_t) \) stopping time \( T \). Working here with the particular \( L = \gamma = \sup\{ s < 1 : B_s = 0 \} \) allows to simplify proofs, and to give closed formulae for the quantities involved.

**Notation:** In this chapter, \( b(\mathcal{P}) \) denotes the space of bounded processes which are predictable with respect to \( (\mathcal{F}_t) \), the Brownian filtration.

### 14.1 Preliminaries and some applications of the balayage formula

#### 14.1.1 A characterization of stopping times.

The following theorem shows, a fortiori, that not all \( (\mathcal{F}_t) \) martingales satisfy (*).

We consider, for the moment, a general filtration \( (\mathcal{G}_t) \), and random time \( R \), and define

\[
\mathcal{G}_R^+ = \sigma\{ z_R, z \text{ any } (\mathcal{G}_t) \text{ progressively measurable process} \},
\]

\[
\mathcal{G}_R^- = \sigma\{ z_R, z \text{ any } (\mathcal{G}_t) \text{ predictable process} \},
\]

Theorem 14.1 (Knight-Maisonneuve [143])

1. If, for all u.i. \( (\mathcal{G}_t) \) martingales \( (X_t) \), one has:

\[
E[X_{\infty}|\mathcal{G}_R^+] = X_R, \quad \text{on } (R < \infty)
\]

then, \( R \) is a \( (\mathcal{G}_t) \) stopping time.

The same statement is valid when \( \mathcal{G}_R^+ \) is replaced by \( \mathcal{G}_R^- \).

Moreover, if \( R \) is a stopping time, \( \mathcal{G}_R = \mathcal{G}_R^+ \).

2. If, for all u.i. \( (\mathcal{G}_t) \) martingales \( (X_t) \), one has:

\[
E[X_{\infty}|\mathcal{G}_R^-] = X_R, \quad \text{on } (R < \infty)
\]

then, \( R \) is a \( (\mathcal{G}_t) \) predictable stopping time.