Progress in Mathematics, Vol. 136

On the cohomology of configuration spaces

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1. Introduction

The aim of this note is to show how previous combinatorial calculations in the computation of the cohomology of configuration spaces can be considerably simplified by more conceptual arguments involving some representation theory. Since I first lectured on these results some other accounts have been given ([CT93, Str93]), partly overlapping with this. Nevertheless, it seemed still worthwile to publish a full account of these considerations.

If Y is any space, we denote by

$$\widetilde{C}_m(Y) = \{(p_1, \ldots, p_m) \in Y^m \mid p_i \neq p_j \text{ for } i \neq j\}$$

the space of m-tuples of pairwise different points in Y. The symmetric group S_m acts on $\widetilde{C}_m(Y)$ in the obvious way; the quotient space

$$C_m(Y) = \widetilde{C}_m(Y)/S_m$$

is the configuration space of m-element subsets of Y. The importance of these spaces lies in their relationship to the theory of iterated loop spaces ([BV68, May72, Seg73, CLM76]).

In the following we shall describe a calculation of the mod p cohomology of the configuration space $C_p(\mathbb{R}^n)$, based on the analysis of the S_p-action on the integral cohomology of $\widetilde{C}_p(\mathbb{R}^n)$. This is one essential step in the general program of computing the cohomology of all $C_m(\mathbb{R}^n)$. This program was carried through by Fred Cohen in [CLM76]. In particular, all the cohomology results are entirely due to him (at least at odd primes); our only contribution consists in supplying more appropriate proofs.

2. The S_m action

We write H^* for cohomology with $\mathbb{Z}$ coefficients and recall first the computation of the cohomology of $\widetilde{C}_m(\mathbb{R}^n)$.

Let $(q_i)_{i \in \mathbb{N}}$ be a fixed sequence of distinct points in $\mathbb{R}^n$ and put $Q_m = \{q_1, \ldots, q_m\}$. We use

$$Q_{m,l} = (q_{l+1}, \ldots, q_{l+m}) \in \widetilde{C}_m(\mathbb{R}^n - Q_l)$$

as the standard base point of the space $\widetilde{C}_m(\mathbb{R}^n - Q_l)$.

For $k < m$ we have a projection $\pi : \widetilde{C}_m(\mathbb{R}^n - Q_l) \to \widetilde{C}_k(\mathbb{R}^n - Q_l)$ given by $\pi(p_1, \ldots, p_m) = (p_1, \ldots, p_k)$. It was shown by Fadell and Neuwirth [FN62] that π is actually a locally trivial fibre bundle. Obviously the fibre $\pi^{-1}Q_{k,l} \subset \widetilde{C}_m(\mathbb{R}^n - Q_l)$ is $\cong \widetilde{C}_{m-k}(\mathbb{R}^n - Q_{k+l})$. We note in passing that these fibrations have sections, defined by adding a fixed configuration of $m-k$ points at a spot far outside the varying configuration of k points.

Now, for $1 \leq i, j \leq m$, $i \neq j$, define $\pi_{ij} : \widetilde{C}_m(\mathbb{R}^n) \to \widetilde{C}_2(\mathbb{R}^n)$ by $\pi_{ij}(p_1, \ldots, p_m) = (p_i, p_j)$. There is an obvious S_2-equivariant homotopy equivalence $S^{n-1} \to \widetilde{C}_2(\mathbb{R}^n)$. Denote by $A \in H^{n-1}(S^{n-1}; \mathbb{Z})$ the standard generator and let

$$A_{i,j} = \pi_{i,j}^*(A) \in H^{n-1}(\widetilde{C}_m(\mathbb{R}^n); \mathbb{Z}) \ .$$

Clearly, one has $A_{j,i} = (-1)^n A_{i,j}$.

Note too, that under restriction to $\widetilde{C}_{m-k}(\mathbb{R}^n - Q_k) \cong \pi^{-1}(Q_k) \subset \widetilde{C}_m(\mathbb{R}^n)$ the classes $A_{i,j}$ with $1 \leq i, j \leq k$ map to zero (since then the map $\pi_{i,j}$ is constant on this space).

Proposition 2.1: *$H^*(\widetilde{C}_{m-k}(\mathbb{R}^n - Q_k); \mathbb{Z})$ is a free abelian group with generators*

$$A_{i_1 j_1} A_{i_2 j_2} \ldots A_{i_s j_s}$$

where

$$k < j_1 < j_2 < \ldots < j_s \leq m \quad \textit{and} \quad i_\nu < j_\nu \quad \textit{for} \quad \nu = 1, \ldots, s \ .$$

Proof: Let $\rho : \widetilde{C}_{m-k}(\mathbb{R}^n - Q_k) \to \mathbb{R}^n - Q_k$ be defined by $\rho(p_1, \ldots, p_{m-k}) = p_1$. Obviously, $\mathbb{R}^n - Q_k$ is homotopy equivalent to a wedge of k spheres S_i^{n-1}, and ρ^* maps the generator of $H^{n-1}(S_i^{n-1})$ to $A_{i,k+1}$. The fibre of ρ is $\widetilde{C}_{m-k-1}(\mathbb{R}^n - Q_{k+1})$; by induction its cohomology is as stated in the proposition. In particular, the inclusion of the fibre is an epimorphism in cohomology. It follows that the spectral sequence collapses (and also that in the case $n = 2$ the cohomology of the fibre is fixed under the fundamental group of the base). □

We state again the most important case $k = 0$. For $n = 2$ this result and the proposition below were first obtained by Arnold [Arn69].

Corollary 2.2: *$H^*(\widetilde{C}_m(\mathbb{R}^n); \mathbb{Z})$ is the free abelian group with generators*

$$A_{i_1 j_1} A_{i_2 j_2} \ldots A_{i_s j_s}$$

where

$$1 < j_1 < \ldots < j_s \leq m \quad \textit{and} \quad 1 \leq i_\nu < j_\nu \quad \textit{for} \quad \nu = 1, \ldots, m \ .$$

In particular, the Poincaré series of $H^*(\widetilde{C}_m(\mathbb{R}^n); \mathbb{Z})$ is

$$\prod_{1 < j \leq m} (1 + (j-1)t^{n-1})$$

and the total rank is $m!$.