Poisson Algebraic Groups and Representations
of Quantum Groups at Roots of 1

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0. Introduction

This paper is an expanded version of the talk I gave at the first ECM held in Paris in July 92. Quantum groups, or better quantized enveloping algebras have been defined around 1985 by Drinfeld and Jimbo, [D], [J], as neither commutative nor cocommutative Hopf algebras obtained by suitably deforming the defining relations of the enveloping algebra of a semisimple group. This new theory has already had spectacular applications to a variety of fields such as the theory of exactly solvable systems (R-matrices) in which Quantum groups were originally introduced, low dimensional topology and Lie theory.

In this paper we shall report on some work developed in collaboration with V. Kac and C. Procesi about unrestricted representations of Quantum groups at roots of unity. The starting point for this research, [DK1] has been to develop this theory in analogy with the representation theory of a semisimple Lie algebra $\mathfrak{g}$, in positive characteristic $p$. When one studies this problem, see for example [WK], one discovers that in the center of the enveloping algebra $U(\mathfrak{g})$ one finds a copy of the symmetric algebra $\text{Symm}(\mathfrak{g})$, and that $U(\mathfrak{g})$ is a finite free module over this symmetric algebra. This implies that $\mathfrak{g}^*$ is, at least roughly a parameter space for the irreducible representations of $\mathfrak{g}$ and furthermore one notices that the set of representations corresponding to points in the same coadjoint orbit are essentially the same. We tried to understand a similar picture in the case of quantized enveloping algebras $U_\varepsilon$ of $\mathfrak{g}$ at a primitive $l$-th root of unity, $\varepsilon$, with some restrictions on $l$ which shall be explained later. And indeed we found many remarkable similarities (I believe that the fact that quantum groups at roots of unity are good liftings of enveloping algebras in positive characteristic is due to G. Lusztig).

Again one finds a large commutative algebra which we call $Z_0$, in the center so that $U_\varepsilon$ is a free finite module over $Z_0$ of rank equal to $l^{\dim \mathfrak{g}}$, so that $\text{Spec}(Z_0)$ is a rough parameter space for the set of irreducible $U_\varepsilon$-modules. At this point one observes that $Z_0$ is a Hopf subalgebra of $U_\varepsilon$ and is closed under the canonical Poisson bracket defined on the center of $U_\varepsilon$. That is $Z_0$ is the coordinate ring of a Poisson group $H$, which is in fact well known and does not depend on $l$. From this one then sees that the role
played in the positive characteristic situation by coadjoint orbits is played here by symplectic leaves in $H$. Indeed one discovers that everything we stated, choosing carefully generators for the quantized enveloping algebra even works when $l$ equals 1, i.e. the deformation parameter is set equal to 1. In this case one obtains the strange fact that the quantized enveloping algebra is also a deformation of the coordinate ring of $H$, a commutative Hopf algebra. This point of view is quite familiar if one studies suitably defined Hopf algebras dual to quantized enveloping algebras, the so called quantum coordinate rings (see for example [S], [LS1]) and holds also for the quantized enveloping algebras themselves as suggested, in a formal setting, in Section 7 of [D] (V. Kac has informed me that our Theorem 3.1. has also been remarked by N. Reshetikhin). Indeed, if one remarks that the enveloping algebra of a Lie algebra $\mathfrak{g}$, is itself a deformation of a Poisson group, namely $\mathfrak{g}^*$ with the usual Kirillov–Kostant Poisson bracket, this observation is not so surprising.

An interesting problem is then of course to understand how general these facts are. Whether and how, for example, we could have started from an arbitrary Poisson group and construct a corresponding Hopf algebra deformation. N. Reshetikhin has recently given a positive answer to this questions at the infinitesimal level. To carry out a program like the one outlined above one needs however more global results (see our definition of a quantum deformation given in Section 2). We propose as a problem to show that any Poisson structure on an algebraic group arising from an algebraic Manin triple (see Section 1 for the definition) admits a quantum deformation in our sense. Here we content ourselves with giving a few more examples other than the one explained above.

Let us review the contents of this paper. In Section 1 we recall the basic definitions and facts about Poisson groups and Manin triples. The only difference with the usual treatments (for example [D], [LW] or [Se]), is that we choose to work in the algebraic category. We also define symplectic leaves and we analyze them in our situation. In Section 2 we define what we mean by a quantum deformation of a Poisson group. Section 3 is essentially devoted to a rather long sketch of the proof that a suitable Hopf algebra over $k[q, q^{-1}]$ contained in the quantized enveloping algebra is a quantum deformation of one of the Poisson groups introduced in Section 1. Section 4 reviews the results to representations at roots of unity. In Section 5 results on the center of quantized enveloping algebras are reviewed, both in the generic case and at roots of unity. Finally Section 6 treats a few more examples, such as the case of the quantum coordinate ring. One word about proofs. This paper contains essentially no proofs except for the sketch of the proof of Theorem 3.1. We have included this since it is the only result