Chapter 9

Compact Operators.

Equations with Compact Operators

In this chapter, we study an important class of linear continuous operators, namely, compact (or completely continuous) operators. On the one hand, compact operators are interesting because they inherit many properties of operators in finite-dimensional spaces. On the other hand, many operators important for applications are compact; in particular, this is true for integral operators with “sufficiently good” kernels.

1 Definition and Properties of Compact Operators

Definition 1.1. Let $E_1$ and $E_2$ be linear normed spaces. A linear operator $A: E_1 \rightarrow E_2$ is called compact if it maps every bounded set in the space $E_1$ into a precompact set of the space $E_2$. The collection of all compact operators acting from $E_1$ into $E_2$ is denoted by $\mathcal{C}(E_1, E_2)$ (or $\mathcal{C}(E)$ if $E_1 = E_2 = E$).

Remark 1.1. Since the operator $A$ is linear, the precompactness of the image of any bounded set follows from the precompactness of the image of the unit ball $B_1(0) \subset E_1$.

Remark 1.2. A compact operator transforms the unit ball $B_1(0)$ into a precompact and, consequently, bounded set. Therefore, any compact operator is bounded, i.e., $\mathcal{C}(E_1, E_2) \subset \mathcal{L}(E_1, E_2)$. At the same time, the property of compactness of linear operators is, generally speaking, stronger than continuity, and this fact is reflected in another name of compact operators — completely continuous operators. The identity operator in an infinite-dimensional space $E$ is an example of a continuous operator which is not compact. Indeed, it maps the unit ball onto itself but $B_1(0)$ is not precompact if dim $E = \infty$.

Remark 1.3. Let $A \in \mathcal{C}(E_1, E_2)$. If $(x_n)_{n=1}^{\infty} \subset E_1$ is a bounded sequence, then the sequence $(Ax_n)_{n=1}^{\infty} \subset E_2$ contains a subsequence convergent in $E_2$. We suggest the reader to verify whether the operator $A: E_1 \rightarrow E_2$ which maps any bounded sequence into a sequence that contains a convergent subsequence is compact.

We consider several examples of compact operators.

Examples

1.1. Let $E_2 = \mathbb{C}^N$. Since any bounded subset of $\mathbb{C}^N$ is precompact, every operator in $\mathcal{L}(E_1, \mathbb{C}^N)$ is compact.

1.2. An operator $A \in \mathcal{L}(E_1, E_2)$ is called finite-dimensional if dim $\mathcal{R}(A) < \infty$. It follows from the previous example that any finite-dimensional operator is compact. As an example of a finite-dimensional operator, one can take an integral operator.
with degenerate kernel acting either on $L_p(R, d\mu)$ or on $C(Q)$ (henceforth, this space is denoted by $E$). A kernel $K$ is called degenerate if it admits a representation in the form of a finite sum

$$K(t, s) = \sum_{i=1}^{n} a_i(t) b_i(s),$$

where $a_i$ and $b_i$ ($i = 1, \ldots, n$) are fixed functions from $E$. It is clear that the range of the integral operator with kernel (1.1) is contained in the linear span of the functions $a_1, \ldots, a_n$, i.e., this operator is finite-dimensional.

1.3. Let $E_1 = E_2 = C([a, b])$ and let $A$ be an integral operator with continuous kernel $K$ (see Example 8.1.3.). We show that $A$ is a compact operator. According to Remark 1.1, it suffices to check whether the set $AB_1(0)$ is precompact. It follows from the boundedness of the operator $A$ that functions belonging to the set $AB_1(0)$ are uniformly bounded. Hence, the first condition of the Arzelà theorem is satisfied. Let us prove that the functions in $AB_1(0)$ are equicontinuous. Since the function $K$ is uniformly continuous in $[a, b] \times [a, b]$, for any $\varepsilon > 0$, one can indicate $\delta > 0$ such that

$$|K(t_1, s) - K(t_2, s)| < \varepsilon (b - a)^{-1}$$

for any $s \in [a, b]$ and all $t_1, t_2 \in [a, b]$ satisfying the inequality $|t_1 - t_2| < \delta$. Thus, for any function $x \in B_1(0)$, we obtain

$$|(Ax)(t_1) - (Ax)(t_2)| \leq \int_{a}^{b} |K(t_1, s) - K(t_2, s)| |x(s)| \, ds < \varepsilon.$$

Therefore, $AB_1(0)$ is a precompact set in $C([a, b])$ and the compactness of the operator $A$ is proved.

**Exercises**

1.1. Prove that

(a) $C(E_1, E_2)$ is a linear space;

(b) for any $A \in C(E_1, E_2)$, $B_1 \in \mathcal{L}(E_1)$, and $B_2 \in \mathcal{L}(E_2)$, we have $AB_1, B_2A \in \mathcal{C}(E_1, E_2)$.

1.2. Prove that an operator of multiplication by an independent variable in $C([0, 1])$ is not compact.

1.3. Let $(\alpha_k)_{k=1}^{\infty}$ be a fixed sequence from $l_\infty$. By using the result of Exercise 7.1.5, prove that the operator $l_p \ni x \mapsto Ax = (\alpha_1 x_1, \alpha_2 x_2, \ldots) \in l_p$ is compact if and only if $\lim_{n \to \infty} \alpha_n = 0$.

Let us study the principal properties of compact operators. For simplicity, we consider only operators acting on a single space.

**Theorem 1.1.** Let $E$ be a Banach space and let $(A_n)_{n=1}^{\infty}$ be a sequence of compact operators that converges to $A$ in the norm of the space $\mathcal{L}(E)$. Then the operator $A$ is compact.