Numerical Methods for Elastic Wave Propagation

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1 Introduction and orientation

There is a great need for numerical methods which treat time-dependent elastic wave propagation problems. Such problems appear in many applications, for example in geophysics or non-destructive testing. In particular, seismic wave propagation is one of the areas where intensive scientific computation has been developed and used since the beginning of the 70’s, with the apparition of finite differences time domain methods (FDTD). Although very old, these methods remain very popular and are widely used for the simulation of wave propagation phenomena or more generally for the numerical resolution of linear hyperbolic systems. They consist in obtaining discrete equations whose unknowns are generally field values at the points of a regular mesh with spatial step $h$ and time step $t$. A prototype of these methods is the famous Yee’s scheme introduced in 1966 for Maxwell’s equations. There are several reasons that explain the success of Yee type schemes, among which their easy implementation and their efficiency which are related to the following properties:

- a uniform regular grid is used for the space discretization, so that there is a minimum of information to store and the data to be computed are structured: in other words, one avoids all the complications due to the use of non uniform meshes.
- an explicit time discretization is applied: no linear system has to be solved at each time step.

For instance, Yee’s scheme is centered, of order two both in space and time, and completely explicit. The stability and accuracy properties of such a scheme are well known (at least in a homogeneous medium in which the classical Fourier analysis can be used). Due to its explicit nature, the scheme is stable under the C.F.L. condition, which says that the ratio $\Delta t/h$ must not exceed a given value. This dictates that the time step cannot be too large, which is not restrictive in practice since a sufficient accuracy requires a small time step.

Of course, in thirty years of research (in the applied mathematics community as well as in the engineering community), considerable progress has been made in the development of more accurate, flexible, and efficient numerical methods for time-dependent

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wave propagation, beginning with higher order finite differences methods. Finite difference methods, although still very popular suffer from a lack of “geometrical flexibility”. This can be also delicate to treat variable coefficients of boundary conditions. That is why, in this course we have chosen to present a class of numerical methods that avoids these drawbacks, numerical methods based on finite elements in space, which authorizes the use on non-regular and unstructured computational grids, and finite differences in time. These are not, of course, the only methods that are able to handle non regular meshes. This is also the case for mixed finite element methods which are variational methods that belong in our sense to the same class of methods as the usual finite element method, even though they have their own interest and their theory is more complicated. Section 3 is really conceived as an introductory course, even though some mathematical background is needed. We shall try to emphasize the main ideas and will present only the more instructive proofs (but bibliographical pointers will be given to complete our presentation). This is the opportunity for us to present in Section 4 some of our recent research in this direction. This is also the occasion to treat the convergence analysis of mixed methods, as a complement to the more detailed analysis made in Section 3.

Our second objective is to present a state of the art in the fundamental transverse question in computational wave propagation: the transparent boundary conditions for bounding artificially a computational domain. This theme is related to the following question: how to reduce to a bounded domain the effective numerical calculation of the solution of a problem which is physically posed in an infinite (or at least very large) domain?

This is once again a vast problem which has retained considerable attention during the past thirty years, in particular from applied mathematicians. We shall focus ourselves to two attractive and competing techniques that have in common the feature of being local in space and time, contrary for instance to some exact methods (Grote and Keller, 1995, 1996) (that we shall not speak about here) no integral operator is involved:

- **Local absorbing boundary conditions.** This type of method has been developed since the late 70’s. The idea is to write on the boundary of the computational domain a boundary condition which is supposed to represent the effect of the presence of the exterior medium. Such a boundary is called local if this boundary condition can be expressed in terms of differential operators (which makes them compatible with traditional discretization techniques). Then, this boundary condition is necessarily not exact and implies some degree of approximation that needs to be quantified in some way. Moreover, the question of the stability of the coupling between the physical propagation interior model and the “unphysical” boundary conditions raises delicate and challenging mathematical questions.

- **Perfectly matched layers.** The idea of an absorbing layer is quite old: the principle is to surround the computational domain with a layer in which the waves are artificially damped. The concept of the perfectly matched layer is much more recent (it has appeared in the middle of the 90’s in electromagnetics). Roughly speaking, the new idea is to build a particular (non-physical) absorbing medium in such a way that no reflection occurs at the interface between the physical medium and the absorbing layer. This technique has generated a considerable literature and is now considered as a very attractive alternative to local absorbing boundary