Diophantine Approximation
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ON THE DIOPHANTINE EQUATION $G_n(x) = G_m(y)$ WITH $Q(x, y) = 0$

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Dedicated to Wolfgang Schmidt on his 70th birthday

1 Introduction

Let $\mathbf{K}$ denote an algebraically closed field of characteristic 0, and let $A_0, \dots, A_{d-1}$, $G_0, \dots, G_{d-1} \in \mathbf{K}[X]$ and $(G_n(X))_{n=0}^{\infty}$ be a sequence of polynomials defined by the d-th order linear recurring relation

$$G_{n+d}(X) = A_{d-1}(X)G_{n+d-1}(X) + \cdots + A_0(X)G_n(X), \quad \text{for } n \geq 0. \tag{1}$$

Furthermore, let $P(X) \in \mathbf{K}[X]$, $\deg P \geq 1$. Recently, we investigated the question, what can be said about the number of solutions of the Diophantine equation

$$G_n(X) = G_m(P(X)). \tag{2}$$

The problem was motivated by properties of families of orthogonal polynomials. For example, the Chebyshev polynomials of the first kind, which are defined by

$$T_n(X) = \cos(n \arccos X),$$

have the well known property that $T_{2n}(X) = T_n(2X^2 - 1)$ for all integers n. Let us mention that all orthogonal polynomials satisfy a second order linear recurring sequence, e.g., for the Chebyshev polynomials we have $T_0(X) = 1$, $T_1(X) = X$ and $T_{n+2}(X) = 2XT_{n+1}(X) - T_n(X)$, $n = 0, 1, 2, \dots$.

Recently, we [8] were able to formulate conditions for sequences of polynomials satisfying a second order linear recurrence under which we could conclude that (2) has

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only finitely many solutions $m, n \in \mathbb{Z}, m, n \geq 0, m \neq n$. For the proof we used the Main Theorem on S-unit equations over finitely generated fields of characteristic zero [3,5]. Furthermore, we were able to quantify our results by transforming our problem in the function field generated by the characteristic root of the recurrence over the rational function field $\mathbf{K}(X)$.

The first author gave suitable extensions of the above results for third order linear recurring sequences (cf. [6]). Later on, we generalized our results to linear recurring sequences $G_n(X)$ of arbitrary order [9]. The conditions are somewhat complicated to be stated, essentially, they ensure that there exist valuations in the underlying function field which have special properties. Let

$$\mathcal{G}(X, T) = T^d - A_{d-1}(X)T^{d-1} - \cdots - A_0(X) \in \mathbf{K}[X][T]$$

denote the characteristic polynomial of the sequence $(G_n(X))_{n=0}^{\infty}$ and $D(X)$ be the discriminant of $\mathcal{G}(X, T)$. We write $\alpha_1, \ldots, \alpha_r$ for the distinct roots of the characteristic polynomial $\mathcal{G}(X, T)$ in the splitting field L of $\mathcal{G}(X, T)$. It is well known that $(G_n(X))_{n=0}^{\infty}$ has a nice "analytic" representation. More precisely, there exist polynomials $C_1(T), \ldots, C_r(T) \in L[T]$ such that

$$G_n(X) = C_1(n)\alpha_1^n + \cdots + C_r(n)\alpha_r^n, \tag{3}$$

holds for all $n \geq 0$. Assuming that $\mathcal{G}(X, T)$ has no multiple roots, i.e., $D(X) \neq 0$, we obtain $C_i(T) = c_i$ for all $i = 1, \ldots, r$, i.e., all the C_i are constant and hence $r = d$. In this case we will call the recurrence $(G_n(X))_{n=0}^{\infty}$ simple (i.e., the characteristic polynomial has only simple roots). Assume now that the d-th order ($d \geq 2$) linear recurring sequence $(G_n(X))_{n=0}^{\infty}$ and the polynomial $P \in \mathbf{K}[X]$ satisfy the following conditions:

(i) None of the roots and the quotients of distinct roots of the characteristic polynomial of $(G_n(X))_{n=0}^{\infty}$ is an element of $\mathbf{K}^*$,
(ii) $\deg P \geq 1$, and $\deg D \geq 1$,
(iii) $\deg A_0 \geq 1$, $R(A_0, \ldots, A_d, G_0, \ldots, G_d) \neq 0$ (for details on the polynomial R we refer to [9]), and
(iv) the set of zeros of A_0 is not equal to that of $A_0(P)$.

Then equation

$$G_n(X) = c\, G_m(P(X)), \tag{4}$$

where $c \in \mathbf{K}^* = \mathbf{K}\backslash\{0\}$ is variable, has at most

$$\begin{aligned} &C(d, A_0, D, P) := \\ &= e^{(6d)^{4d}} \left(\log\left(d^{2d^2} \deg D(\deg P + 1)\right)\right)^{2d^2} (2ed)^{30d^3 d!^2 \deg A_0 \deg P} \end{aligned}$$

solutions $(n, m) \in \mathbb{Z}^2$ with $n, m \geq 0, n \neq m$. We also obtained the result under different conditions (see [9, Theorem 2.3]). For the special case of equation (2) we could even show more: assuming the conditions from above with

(i′) None of the roots and the quotients of distinct roots of the characteristic polynomial of $(G_n(X))_{n=0}^{\infty}$ is a root of unity,