1 Introduction

Polylogarithmic functions are defined by series

$$L_k(z) = \sum_{\nu=1}^{\infty} \frac{z^\nu}{\nu^k}, \quad k \geq 1.$$  

Due to equalities $L_k(1) = \zeta(k)$, $k \geq 2$, they play an important role in study of arithmetic properties of Riemann zeta-function $\zeta(s)$ at integer points.

More generally for any rational function $R(s)$ that can be presented as a sum of simple fractions

$$R(s) = \sum_{\ell \in \mathcal{P}} \sum_{k=1}^{d(\ell)} \frac{B_{\ell, k}}{(s + \ell)^k}, \quad B_{\ell, k} \in \mathbb{Q},$$  

where $\mathcal{P}$ is a set of distinct positive integers and $d(\ell) \geq 0$, one can find the following equalities:

$$F(z) = \sum_{\nu=0}^{\infty} R(\nu) z^\nu = \sum_{\ell \in \mathcal{P}} \sum_{k=1}^{d(\ell)} B_{\ell, k} \sum_{\nu=0}^{\infty} \frac{z^\nu}{(\nu + \ell)^k} =$$

$$= \sum_{\ell \in \mathcal{P}} \sum_{k=1}^{d(\ell)} B_{\ell, k} z^{-\ell} \sum_{\nu=\ell}^{\infty} \frac{z^\nu}{\nu^k} = \sum_{\ell \in \mathcal{P}} \sum_{k=1}^{d(\ell)} B_{\ell, k} z^{-\ell} \left( L_k(z) - \sum_{\nu=1}^{\ell-1} \frac{z^\nu}{\nu^k} \right).$$

This confirms that the function $F(z)$ is a linear form in 1 and polylogarithms with coefficients in $\mathbb{Q}[1/z]$. It is clear that analogous result can be proved if we put as coefficients of the series $F(z)$ any derivative of $R(s)$ and shift the lower limit of summation on any admissible integer number. The following proposition defines the general construction.
**Proposition 1.** For any complex \( z \) from the convergence domain of the series

\[
G_r(z) = \frac{(-1)^{r-1}}{(r-1)!} \sum_{v=1}^{\infty} R^{(r-1)}(v-a)z^v
\]

the following identity holds

\[
G_r(z) = A_0(z^{-1}) + \sum_{k=1}^{q} A_k(z^{-1}) L_{k+r-1}(z).
\]

Here \( q = \max_{\ell \in \mathcal{P}} d(\ell) \) and

\[
A_k(x) = \left( \frac{k+r-2}{r-1} \right)^{k} \sum_{\ell \in \mathcal{P}, d(\ell) \geq k} B_{\ell,k} x^{\ell-a}, \quad k = 1, \ldots, q,
\]

\[
A_0(x) = - \sum_{\ell \in \mathcal{P}} \sum_{k=1}^{d(\ell)} \sum_{v=1}^{\ell-a} \left( \frac{k+r-2}{r-1} \right) B_{\ell,k} v^{1-k-r} x^{\ell-a-v}.
\]

**Proof.** See [9, Proposition 1].

For arithmetic applications of this construction one has to choose the rational function \( R(s) \) in such a way that the number \( G_r(1) \), a linear form in zeta-values, be rather small, and the coefficients \( A_k(1) \in \mathbb{Q} \) have common denominator and magnitude that are not very large. In most cases the choice of the function \( R(s) \) may be described as follows.

Let \( a_j > 1, b_j > 1, j = 1, \ldots, m, \) be integers. Define

\[
R(s) = \gamma \prod_{j=1}^{m} \frac{\Gamma(s+a_j)}{\Gamma(s+b_j)} \in \mathbb{Q}(s),
\]

where \( \gamma \) is a rational number that will be defined later.

With the choice (5) there exist integral representations for \( G_r(z), r \geq 1 \) that are useful in applications for the computation of the asymptotic of the constructed linear forms.

Let \( r \geq 1 \) be integer and \( u \) be a complex number. Write

\[
I_r(u) = \frac{1}{2\pi i} \int_{L} R(s) \left( \frac{\pi}{\sin \pi s} \right)^r e^{\pi ius} ds,
\]

where the path of integration \( L \) goes from \(-i\infty\) to \( i\infty\) and separates the poles of \( \Gamma(s+a_j), 1 \leq j \leq m, \) from points 0, 1, 2, \ldots.

It is easy to check, that the integral (6) converges for \( |\Re u| < r \). In the following proposition we express the function \( G_r(z) \) in terms of integrals (6). For simplicity we assume that \( a = 0 \).

**Proposition 2.** Let \( r \) be integer, \( r \geq 1 \), and the rational function \( R(s) \) is defined by (1).