Diophantine Approximation
H. P. Schlickewei et al., Editors

METRIC DISCREPANCY RESULTS FOR SEQUENCES $\{n_k x\}$ AND DIOPHANTINE EQUATIONS

István Berkes[1], Walter Philipp[2], and Robert F. Tichy[3]
[1] *Institut für Statistik, Technische Universität Graz, Steyrergasse 17, 8010 Graz, Austria*
berkes@tugraz.at
[2] *Department of Statistics, University of Illinois at Urbana-Champaign, Champaign, Illinois, USA*
[3] *Institut für Mathematik A, Technische Universität Graz, Steyrergasse 30, 8010 Graz, Austria*
tichy@tugraz.at

Dedicated to Professor Wolfgang M. Schmidt on the occasion of his 70th birthday.

1 Introduction

Let (n_k) be an increasing sequence of positive integers. For $0 \leq x \leq 1$, set

$$\eta_k = \eta_k(x) := n_k x \pmod 1 . \tag{1}$$

The discrepancy of the first N elements of the sequence (η_k) is defined as

$$D_N = D_N(x) := \sup_{0 \leq s \leq 1} \left| \frac{1}{N} \operatorname{card}(k \leq N : \eta_k(x) \leq s) - s \right| . \tag{2}$$

In his fundamental paper on uniform distribution mod 1, H. Weyl [23] proved, among many other things, that $D_N(x) \to 0$ for almost all $x \in (0, 1)$, i.e., that $(n_k x)$ is uniformly distributed mod 1 for all $x \in (0, 1)$ except for a set of Lebesgue measure zero. This was later improved independently by Cassels [5] and by Erdős and Koksma [7] who proved that for almost all $x \in (0, 1)$

$$N D_N(x) = O(N^{1/2}(\log N)^{5/2+\varepsilon}), \qquad \varepsilon > 0.$$

The best result so far has been achieved by R. C. Baker [1], who reduced the exponent $\frac{5}{2}$ of the logarithm to $\frac{3}{2}$. The exact exponent of the logarithm is still an open problem, except for the fact that it cannot be less than $\frac{1}{2}$, as was shown by Berkes and Philipp [4].

Keywords. Discrepancy, lacunary sequences, Diophantine equations, law of iterated algorithm.

2000 Mathematics subject classification. 11K38, 11D45, 42A55, 60F15.

Determining the exact order of magnitude of $D_N(x)$ for a concrete sequence (n_k) is generally a hard problem and a satisfactory solution exists only in a few special cases. For $n_k = k$, Kesten [15] proved that

$$N D_N(x) \sim \frac{2}{\pi^2} \log N \log\log N$$

in measure. (For the remainder term, see Schoissengeier [20].) Another important case when a sharp bound for the magnitude of $D_N(x)$ is known is the case when (n_k) is a lacunary sequence. Philipp [17,18] proved that if (n_k) satifies the Hadamard gap condition

$$n_{k+1}/n_k \geq 1 + \rho, \quad \rho > 0,\ k = 1, 2, \ldots, \tag{3}$$

then we have for almost all x

$$\frac{1}{4} \leq \limsup_{N\to\infty} \frac{N D_N(x)}{\sqrt{N \log\log N}} \leq C(\varrho), \tag{4}$$

where $C(\varrho) \ll \frac{1}{\varrho}$. This result has an obvious probabilistic flavor. If (x_n) is a sequence of independent random variables uniformly distributed (in the probabilistic sense) over $(0, 1)$, then by the classical Chung–Smirnov law of the iterated logarithm for empirical distribution functions (see, e.g., Shorack and Wellner [21, p. 504]), the discrepancy D_N^* of $(x_n, n \leq N)$ satisfies

$$\limsup_{N\to\infty} \frac{N D_N^*}{\sqrt{N \log\log N}} = \frac{1}{\sqrt{2}} \tag{5}$$

with probability one. Thus, roughly speaking, under the Hadamard gap condition (3) the sequence $n_k x$ (mod 1) behaves like a sequence of independent random variables. This heuristics, which plays an important role in harmonic analysis and is the key for understanding a number of interesting phenomena (see, e.g., Kac [13]), should be used, however, with great care. Berkes and Philipp [4] have constructed sequences (n_k) satisfying (3) for which the lower bound $\frac{1}{4}$ in (4) can be improved to $c \log\log \frac{1}{\varrho}$ with an absolute constant c. Hence there cannot be an upper bound (4), independent of ϱ, that works for all sequences (n_k) satisfying a Hadamard gap condition (3). A deeper analysis of the problem shows that under the Hadamard gap condition (3), the behavior of D_N is determined by a delicate interplay between the speed of growth and the number-theoretic properties of (n_k). If (n_k) grows extremely rapidly, then $\{n_k x\}$ is indeed a nearly i.i.d. (independent, identically distributed) sequence of random variables as one can easily see from the mixing relation

$$\lim_{n\to\infty} |x \in (\alpha, \beta) : \{nx\} \leq t| = (\beta - \alpha)t \qquad (0 \leq \alpha < \beta \leq 1),$$

where $|\cdot|$ stands for the Lebesgue measure. See Philipp [18, Lemma 4.2.1], where a remainder term is also given. Specifically, if

$$\sum_{k=1}^{\infty} n_k/n_{k+1} < \infty,$$

then the limsup in (4) is $1/\sqrt{2}$, in accordance with (5). (This follows easily using the approximation method in Berkes [2].) If, however, n_{k+1}/n_k is bounded, then the arithmetic structure of (n_k) comes into play. The number-theoretic effect becomes