The Equation $\mathcal{I}(\mathcal{S}(x, y), z) = \mathcal{T}(\mathcal{I}(x, z), \mathcal{I}(y, z))$
for t-representable t-conorms and t-norms
Generated from Continuous, Archimedean Operations

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Abstract. In this article we continue investigations presented at previous WILF 2011 conference which are connected with distributivity of implication operations over t-representable t-norms and t-conorms. Our main goal is to show the general method of solving the following distributivity equation $\mathcal{I}(\mathcal{S}(x, y), z) = \mathcal{T}(\mathcal{I}(x, z), \mathcal{I}(y, z))$, when $\mathcal{S}$ is a t-representable t-conorm on $\mathcal{L}^I$ generated from two continuous, Archimedean t-conorms, $\mathcal{T}$ is a t-representable t-norm on $\mathcal{L}^I$ generated from two continuous, Archimedean t-norms and $\mathcal{I}$ is an unknown function.

Keywords: Interval-valued fuzzy sets, Triangular norm, Triangular conorm, Distributivity equations, Functional equations.

1 Introduction

Distributivity of (classical) fuzzy implications over different fuzzy logic connectives has been studied in the recent past by many authors (see chronologically [2], [32], [12], [29], [30], [11],[3],[8]). These equations have a very important role to play in efficient inferencing in approximate reasoning, especially in fuzzy control systems. Given an input “$\tilde{x}$ is $A$”, the role of an inference mechanism is to obtain a fuzzy output $B'$ that satisfies some desirable properties. The most important inference schemas are fuzzy relational inference and similarity based reasoning. In the first case the inferred output $B'$ is obtained either as

(i) sup $- T$ composition, as in the compositional rule of inference (CRI) of Zadeh (see [33]), or
(ii) inf $- I$ composition, as in the Bandler-Kohout Subproduct (BKS) (see [13]),
of $A'$ and given rules. Since all the rules of an inference engine are exercised during every inference cycle, the number of rules directly affects the computational duration of the overall application.
To reduce the complexity of fuzzy “IF-THEN” rules, Combs and Andrews [16] proposed an equivalent transformation of the CRI to mitigate the computational cost. In fact, they required of the following classical tautology

\[(p \land q) \rightarrow r = (p \rightarrow r) \lor (q \rightarrow r).\]

so we see that the distributivity of fuzzy implications over t-norms (or t-conorms) play a major role in this transformation. Subsequently, there were many discussions (see [14], [15], [20], [28]), most of them pointed out the need for a theoretical investigation required for employing such equations. Later, the similar method but for similarity based reasoning was demonstrated by Jayaram [24]. For an overview of the most important methods that reduce the complexity of different inference systems and concrete examples see [10, Chapter 8].

Recently, in [4], [5], [6] (for the full article see [9]) and [7] we have discussed the distributivity equation of implications

\[I(x, T(y, z)) = T(I(x, y), I(x, z))\]

over t-representable t-norms generated from continuous Archimedean t-norms, in interval-valued fuzzy sets theory. In these articles we have obtained the solutions for each of the following functional equations, respectively:

\[
\begin{align*}
 f(u_1 + v_1, u_2 + v_2) &= f(u_1, u_2) + f(v_1, v_2), \\
 g(\min(u_1 + v_1, a), \min(u_2 + v_2, a)) &= g(u_1, u_2) + g(v_1, v_2), \\
 h(\min(u_1 + v_1, a), \min(u_2 + v_2, a)) &= \min(h(u_1, u_2) + h(v_1, v_2), b), \\
 k(u_1 + v_1, u_2 + v_2) &= \min(k(u_1, u_2) + k(v_1, v_2), b),
\end{align*}
\]

where \(a, b > 0\) are fixed real numbers, \(f: L^\infty \to [0, \infty], g: L^a \to [0, \infty], h: L^a \to [0, b],\) and \(k: L^\infty \to [0, b]\) are unknown functions. The above we use the following notation

\[
L^\infty = \{(u_1, u_2) \in [0, \infty]^2 \mid u_1 \geq u_2\},
\]

\[
L^a = \{(u_1, u_2) \in [0, a]^2 \mid u_1 \geq u_2\}.
\]

More precisely, the solutions of Eq. (A) have been presented in [4] Proposition 3.2], the solutions of Eq. (B) have been presented in [5] Proposition 4.2], the solutions of Eq. (C) have been presented in [9] Proposition 5.2] and the solutions of Eq. (D) have been presented in [7] Proposition 3.2].

In this paper we continue these investigations, but for the following functional equation

\[I(\mathcal{S}(x, y), z) = \mathcal{T}(I(x, z), I(y, z)),\]

satisfied for all \(x, y, z \in \mathcal{L}^I\), when \(\mathcal{S}\) is a t-representable t-conorm on \(\mathcal{L}^I\) generated from two continuous, Archimedean t-conorms \(S_1, S_2\), \(\mathcal{T}\) is a t-representable t-norm on \(\mathcal{L}^I\) generated from two continuous, Archimedean t-norms \(T_1, T_2\) and \(I\) is an unknown function.

Please note that the solutions for this Eq. (D-ST) in the classical case, i.e. for classical continuous Archimedean t-norms and t-conorms have been presented by the author in [8].