Belief Merging
in Dynamic Logic of Propositional Assignments

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Abstract. We study syntactical merging operations that are defined semantically by means of the Hamming distance between valuations; more precisely, we investigate the \(\Sigma\)-semantics, Gmax-semantics and max-semantics. We work with a logical language containing merging operators as connectives, as opposed to the metalanguage operations of the literature. We capture these merging operators as programs of Dynamic Logic of Propositional Assignments DL-PA. This provides a syntactical characterisation of the three semantically defined merging operators, and a proof system for DL-PA therefore also provides a proof system for these merging operators. We explain how PSPACE membership of the model checking and satisfiability problem of star-free DL-PA can be extended to the variant of DL-PA where symbolic disjunctions that are parametrised by sets (that are not defined as abbreviations, but are proper connectives) are built into the language. As our merging operators can be polynomially embedded into this variant of DL-PA, we obtain that both the model checking and the satisfiability problem of a formula containing possibly nested merging operators is in PSPACE.

Keywords: belief merging, belief change, dynamic logic.

1 Introduction

To merge a vector of belief bases \(E = \langle B_1, \cdots, B_n \rangle\) means to build a new belief base \(\Delta(E)\). In the literature, \(E\) is called a profile, and \(\Delta(E)\) is sometimes called the fusion of \(E\). Much efforts were spent on the characterisation of ‘good’ merging operations \(\Delta\) by means of rationality postulates \([14]–[16]\). Beyond such families of abstract belief merging operations satisfying the postulates, several concrete operations were also introduced and studied in the literature. Some are syntax-based and others are semantic. The former are also called ‘formula-based’, and the latter are called ‘model-based’ or ‘distance-based’. An example of the former is the MCS operation \([2]\), where each element \(B_i\) of \(E\) is viewed as a set of formulas that is not closed under logical consequence and where the construction of \(\Delta(E)\) is based on the extraction of maximal consistent subsets of each \(B_i\) of \(E\). Such operations are syntax dependent: they do not guarantee that the merging of logically equivalent profiles leads to merged bases that are logically equivalent\(^1\).

\(^1\) Two profiles \(E\) and \(E'\) are logically equivalent if for every \(B_i\) in \(E\) there is a logically equivalent \(B'_i\) in \(E'\) and the other way round, for every \(B'_i\) in \(E'\) there is a logically equivalent \(B_j\) in \(E\).
In contrast, syntax independence is guaranteed by the semantic merging operations, whose most prominent are $\Delta_\Sigma$, $\Delta_{\text{max}}$, and $\Delta_{\text{Gmax}}$ [19][20]. These operations work on valuations of classical propositional logic. Indeed, even when the elements of the input profile are presented as formulas or sets thereof, the merging procedure starts by computing their models. The output set of valuations is sometimes transformed into a formula characterising the set, which can always be done because these operations are presented in terms of a finite set of propositional variables.

Contrasting with the existing literature, the present paper studies concrete semantic merging operations from a syntactic perspective: given a vector of formulas $E$, our aim is to obtain a syntactical representation of the merged belief base $\Delta(E)$, for $\Delta$ being $\Delta_\Sigma$, $\Delta_{\text{max}}$, or $\Delta_{\text{Gmax}}$. As we have already said above, when the language is finite then it is easy to construct a formula representing $\Delta(E)$: it suffices to take the disjunction of the formulas describing the models of $\Delta(E)$, where each of these model descriptions is a conjunction of literals. Is there a better, more direct way of building a syntactic representation? In this paper we propose a powerful yet simple logical framework: Dynamic Logic of Propositional Assignments, abbreviated $\text{DL-PA}$ [1]. $\text{DL-PA}$ is a simple instantiation of Propositional Dynamic Logic PDL [7, 8]. Just as PDL, its language is built with two ingredients: atomic formulas and atomic programs. In both logics, atomic formulas are propositional variables. While PDL has abstract atomic programs, the atomic programs of $\text{DL-PA}$ are assignments of propositional variables to either true or false, respectively noted $p \leftarrow \top$ and $p \leftarrow \bot$. The assignment $p \leftarrow \top$ corresponds to an update by $p$, while the assignment $p \leftarrow \bot$ corresponds to an update by $\neg p$. Complex programs $\pi$ are built from atomic programs by the standard PDL program operators of sequential composition, nondeterministic composition, finite iteration (the so-called Kleene star), and test. Just as for PDL, $\text{DL-PA}$ has formulas of the form $\langle \pi \rangle \varphi$ and $[\pi] \varphi$, where $\pi$ is a program and $\varphi$ is a formula. The former expresses that $\varphi$ is true after some possible execution of $\pi$, and the latter expresses that $\varphi$ is true after every possible execution of $\pi$. For example, the $\text{DL-PA}$ formula $\langle p \leftarrow \top \cup p \leftarrow \bot \rangle \varphi$ captures the propositional quantification $\exists p. \varphi$, illustrating that $\text{DL-PA}$ naturally captures Quantified Boolean Formulas (QBF).

It is shown in [1] that $\text{DL-PA}$ formulas can be reduced to equivalent Boolean formulas. Just as for QBFs, the original formula is more compact than the equivalent Boolean formula. Star-free $\text{DL-PA}$ has the same mathematical properties as the QBF reasoning problems; in particular, model checking, satisfiability and validity are all PSPACE complete. We believe $\text{DL-PA}$ to be a more natural and flexible tool than QBF to reason about domains involving dynamics due to its more elaborate account in terms of programs.

Our main contributions are polynomial embeddings of semantic belief merging operators into $\text{DL-PA}$: to every profile $E$ and merging operation $\Delta$ we associate a $\text{DL-PA}$ formula $\varphi(\Delta, E)$, and we prove that the merged profile $\Delta(E)$ has the same models as $\varphi(\Delta, E)$. Then $\varphi(\Delta, E)$ may then be reduced to a Boolean formula, thus providing a syntactical representation of $\Delta(E)$ in propositional logic. A further contribution of our paper is a presentation of merging in terms of a recursive language with several merging operators $\Delta^\sigma$ in the object language, one operator per semantics $\sigma$. This contrasts with the usual presentations in terms of metalanguage operations (where we systematically use the term operator for connectives in the object language, while we reserve the term operation for functions from the metalanguage).