Two-Term Disjunctions on the Second-Order Cone

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Abstract. Balas introduced disjunctive cuts in the 1970s for mixed-integer linear programs. Several recent papers have attempted to extend this work to mixed-integer conic programs. In this paper we develop a methodology to derive closed-form expressions for inequalities describing the convex hull of a two-term disjunction applied to the second-order cone. Our approach is based on first characterizing the structure of undominated valid linear inequalities for the disjunction and then using conic duality to derive a family of convex, possibly nonlinear, valid inequalities that correspond to these linear inequalities. We identify and study the cases where these valid inequalities can equivalently be expressed in conic quadratic form and where a single inequality from this family is sufficient to describe the convex hull. Our results on two-term disjunctions on the second-order cone generalize related results on split cuts by Modaresi, Kılınç, and Vielma, and by Andersen and Jensen.

Keywords: Mixed-integer conic programming, second-order cone programming, cutting planes, disjunctive cuts.

1 Introduction

A mixed-integer conic program is a problem of the form

$$\sup\{d^\top x : \; Ax = b, \; x \in \mathbb{K}, \; x_j \in \mathbb{Z} \; \forall j \in J\}$$

where $\mathbb{K} \subset \mathbb{R}^n$ is a regular (full-dimensional, closed, convex, and pointed) cone, A is an $m \times n$ real matrix of full row rank, d and b are real vectors of appropriate dimensions, and $J \subseteq \{1, \ldots, n\}$. Mixed-integer conic programming (MICP) models arise naturally as robust versions of mixed-integer linear programming (MILP) models in finance, management, and engineering [1]. MILP is the special case of MICP where $\mathbb{K}$ is the nonnegative orthant, and it has itself numerous applications. A successful approach to solving MILP problems has been to first solve the continuous relaxation, then add cuts, and finally perform branch-and-bound using this strengthened formulation. A powerful way of generating such cuts is to impose a valid disjunction on the continuous relaxation and to generate tight convex inequalities for the resulting disjunctive set. Such inequalities are known as *disjunctive cuts*. Specifically, the integrality

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conditions on the variables x_j, $j \in J$, imply linear *two-term disjunctions* of the form $\pi^\top x \le \pi_0 \vee \pi^\top x \ge \pi_0 + 1$ where $\pi_0 \in \mathbb{Z}$, $\pi_j \in \mathbb{Z}$, $j \in J$, and $\pi_j = 0$, $j \notin J$. Following this approach, the feasible region for MICP problems can be relaxed to $\{x \in \mathbb{K} : Ax = b, \pi^\top x \le \pi_0 \vee \pi^\top x \ge \pi_0 + 1\}$. More general two-term disjunctions arise in complementarity [2] and other non-convex optimization [3] problems. Therefore, it is interesting to study relaxations of MICP problems of the form

$$\sup\{d^\top x : x \in C_1 \cup C_2\} \quad \text{where}$$
$$C_i := \{x \in \mathbb{K} : Ax = b, \ c_i^\top x \ge c_{i,0}\} \quad \text{for} \quad i \in \{1, 2\} \tag{1}$$

and to derive strong valid inequalities for the convex hull $\text{conv}(C_1 \cup C_2)$, or the closed convex hull $\overline{\text{conv}}(C_1 \cup C_2)$. When $\mathbb{K}$ is the nonnegative orthant, Bonami et al. [4] characterize $\overline{\text{conv}}(C_1 \cup C_2)$ by a finite set of linear inequalities. The purpose of this paper is to provide closed-form expressions for convex inequalities describing $\overline{\text{conv}}(C_1 \cup C_2)$ for other cones such as the second-order (Lorentz) cone $\mathbb{K}_2^n := \{x \in \mathbb{R}^n : \|\tilde{x}\|_2 \le x_n\}$ where $\tilde{x} := (x_1, \ldots, x_{n-1})$. We first review related results from the literature.

Disjunctive cuts were introduced by Balas [5] for MILP in the early 1970s. *Chvátal-Gomory*, *lift-and-project*, *mixed-integer rounding (MIR)*, and *split cuts* are all special types of disjunctive cuts. Recent efforts on extending the cutting plane theory for MILP to the MICP setting include the work of Çezik and Iyengar [6] for Chvatal-Gomory cuts, Stubbs and Mehrotra [7], Drewes [8], and Bonami [9] for lift-and-project cuts, and Atamtürk and Narayanan [10] for MIR cuts. Kılınç-Karzan [11] analyzed properties of minimal valid linear inequalities for general conic sets with a disjunctive structure and showed that these are sufficient to describe the closed convex hull. Such general sets from [11] include two-term disjunctions on the cone $\mathbb{K}$ considered in this paper. Bienstock and Michalka [12] studied the characterization and separation of valid linear inequalities that convexify the epigraph of a convex, differentiable function restricted to a non-convex domain. In the last few years, there has been growing interest in developing closed-form expressions for convex inequalities that fully describe the convex hull of a disjunctive conic set. Dadush et al. [13] and Andersen and Jensen [14] derived split cuts for ellipsoids and the second-order cone, respectively. Modaresi et al. [15] extended this work to essentially all cross-sections of the second-order cone. Belotti et al. [16] identified a procedure for constructing two-term disjunctive cuts under the assumptions that $C_1 \cap C_2 = \emptyset$ and the sets $\{x \in \mathbb{K} : Ax = b, c_1^\top x = c_{1,0}\}$ and $\{x \in \mathbb{K} : Ax = b, c_2^\top x = c_{2,0}\}$ are bounded.

In this paper we study general two-term disjunctions on conic sets and give closed-form expressions for the tightest disjunctive cuts that can be obtained from these disjunctions in a large class of instances. We focus on the case where C_1 and C_2 in (1) above have an empty set of equations $Ax = b$. That is to say, we consider

$$C_1 := \{x \in \mathbb{K} : c_1^\top x \ge c_{1,0}\} \quad \text{and} \quad C_2 := \{x \in \mathbb{K} : c_2^\top x \ge c_{2,0}\}. \tag{2}$$