Chapter 15
Multidimensional Wave Problems
in Layered Random Media

Consider now extensions of the stationary problem on plane waves in randomly layered media to the simplest multidimensional problems. Among these are the nonstationary problems on propagation of time-domain impulses in randomly layered media and the three-dimensional steady-state problem on the field of a point source in layered media.

15.1 Nonstationary Problems

15.1.1 Formulation of Boundary-Value Wave Problems

Consider the nonstationary problem on plane wave \( f[t + (x - L)/c_0] \) \((c_0\) is the velocity of the wave in free space) incident from region \(x > L\) on medium layer occupying the portion of space \(L_0 < x < L\). The wavefield in the layer satisfies the wave equation with attenuation \(\tilde{\gamma}\).

\[
\left[ \frac{\partial^2}{\partial x^2} - \frac{1}{c^2(x)} \frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} + \tilde{\gamma} \right) \right] u(x, t) = 0 \quad (15.1)
\]

with the boundary conditions

\[
\left. \left( \frac{\partial}{\partial x} + \frac{\partial}{c_0 \partial t} \right) u(x, t) \right|_{x=L} = \left. \frac{2}{c_0} \frac{\partial}{\partial t} f(t) \right|_{x=L}, \quad \left. \left( \frac{\partial}{\partial x} - \frac{\partial}{c_0 \partial t} \right) u(x, t) \right|_{x=L_0} = 0. \quad (15.2)
\]

Similarly, for the plane wave source located at point \(x_0\) in the medium, we have the boundary-value problem
\[
\left[ \frac{\partial^2}{\partial x^2} - \frac{1}{c^2(x)} \frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} + \tilde{\gamma} \right) \right] u(x, x_0; t) = -\frac{2}{c_0} \delta(x - x_0) \frac{\partial}{\partial t} f(t),
\]

\[
\left. \left( \frac{\partial}{\partial x} + \frac{\partial}{c_0 \partial t} \right) u(x, x_0; t) \right|_{x=L} = 0, \quad \left. \left( \frac{\partial}{\partial x} - \frac{1}{c_0 \partial t} \right) u(x, x_0; t) \right|_{x=L} = 0.
\]

(15.3)

Note that boundary-value problem (15.1), (15.2) coincides with boundary-value problem (15.3) for the source located at layer boundary, i.e., at \( x_0 = L \). In this case, we have \( u(x, L; t) = u(x; t) \).

The solution to problem (15.3) can be represented in the form of the Fourier integral (parameter \( \tilde{\gamma} \) is assumed small)

\[
u(x, x_0; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega G_\omega(x, x_0)f(\omega)e^{-i\omega t}, \quad G_\omega(x, x_0) = \int_{-\infty}^{\infty} dt G(x, x_0; t)e^{i\omega t},
\]

(15.4)

where \( f(\omega) = \int_{-\infty}^{\infty} dt f(t)e^{i\omega t} \).

Function \( G_\omega(x, x_0) \) is the solution of the stationary problem on the field of the point source in randomly layered medium (11.12)

\[
\frac{d^2}{dx^2} G_\omega(x, x_0) + k^2 [1 + \varepsilon(x)] G_\omega(x, x_0) = 2ik\delta(x - x_0),
\]

(15.5)

where

\[
\frac{1}{c^2(x)} = \frac{1}{c_0^2} [1 + \varepsilon(x)], \quad \varepsilon(x) = \varepsilon_1(x) + \tilde{\gamma} \frac{i}{\omega}, \quad k = \frac{\omega}{c_0}.
\]

We considered this problem earlier. Parameter \( \tilde{\gamma} \) characterizes wave absorption in the medium and is related to parameter \( \gamma \) introduced earlier through the relationship \( \gamma = \tilde{\gamma}/2c_0 \).

Introduce Green’s nonstationary function \( G(x, L; t) \). At the boundary \( x = L \), wave \( f[t + (x - L)/c_0] \) incident on the layer creates the distribution of sources \( \tilde{f}(t_0) \) such that

\[
f(t) = \frac{1}{2c_0} \int_{-\infty}^{\infty} dt_0 \theta(t - t_0) \tilde{f}(t_0), \quad \tilde{f}(t_0) = 2c_0 \frac{\partial}{\partial t_0} f(t_0).
\]

Then, we can represent the wavefield in the layer in the form