Chapter 7
Diffusion and Clustering of Settling Tracer in Random Flows

7.1 State of Art and Main Equation of the Problem

Dynamics of foreign particles and inclusions whose velocity relative resting (on average) medium is quite appreciable in view of buoyancy and gravity forces in hydrodynamic flows attracts attention of researchers beginning from the classical paper by Stokes (1851) \[304\]. Importance of these investigations follows from urgency of the problem in the context of different ecological and climatological problems in the Earth atmosphere and ocean. Among the inclusions mentioned are the fine dust injected by industrial objects and sources of ecocatastrophes, artificial condensation centers, and artificial centers of scattering.

Diffusion of the number density field \(n(r,t)\) of these inertia particles (the number of particles per unit volume) and the density \(\rho(r,t) = \rho_0 n(r,t)\) of passive tracer moving in random hydrodynamic flows described by the Eulerian velocity field \(u(r,t)\) satisfies, as earlier, the continuity equations (3.1), (3.2), and the field of particle velocities \(V(r,t)\) in hydrodynamic flow satisfies, in the case of low-inertia particles, the quasi-linear partial differential equation (3.3), page 28.

We neglect the effect of molecular diffusion, which is valid during the initial stages of diffusion. This effect is described by Eq. (3.9), page 30 and must be taken into consideration on later stages of temporal evolution.

In the case of low-inertia particles under the action of buoyancy and gravity forces, the field of particle velocities \(V(r,t)\) in random hydrodynamic flow \(u(r,t)\) can be described by the quasi-linear partial differential equation

\[
\left( \frac{\partial}{\partial t} + V(r,t) \frac{\partial}{\partial r} \right) V(r,t) = -\lambda [V(r,t) - u(r,t)] + g \left( 1 - \frac{\rho_0}{\rho_p} \right), \tag{7.1}
\]

rather than by Eq. (3.3). Here \(g\) is the gravity acceleration and \(\rho_p\) and \(\rho_0\) are the densities of particles and medium respectively. As earlier, we will consider this equation as the phenomenological equation.
Settling velocity $v$ of floating-up tracer is usually directed along the vertical; it is formed by the balance of the buoyancy and viscous friction forces acting on moving tracer and is given by the formula

$$
\frac{g}{\lambda} \left( 1 - \frac{\rho_0}{\rho_p} \right) = v.
$$

Setting

$$
V(r, t) = v + v(r, t),
$$

where $v(r, t)$ is the fluctuation of the tracer velocity field relative to $v$, we can rewrite the system of equations (3.1), page 28 and (7.1) in the form

$$
\left( \frac{\partial}{\partial t} + [v + v(r, t)] \frac{\partial}{\partial r} \right) \rho(r, t) = -\frac{\partial v(r, t)}{\partial r} \rho(r, t), \quad \rho(r, 0) = \rho_0(r),
$$

(7.2)

$$
\left( \frac{\partial}{\partial t} + [v + v(r, t)] \frac{\partial}{\partial r} \right) v(r, t) = -\lambda [v(r, t) - u(r, t)].
$$

(7.3)

In the general case, velocity field $u(r, t)$ is assumed to be the divergent Gaussian random field statistically homogeneous and isotropic in space and stationary in time; we will assume that $\langle u(r, t) \rangle = 0$, and the corresponding correlation and spectral tensors are given by Eqs. (4.1), (4.4), page 39.

As earlier, we will assume that variance $\sigma_u^2 = \langle u^2(r, t) \rangle$ is sufficiently small and defines the main small statistical parameter of the problem.

### 7.1.1 Particle Diffusion (Lagrangian Description)

Equation for tracer velocity field $v(r, t)$ (7.3) is the first order partial differential equation (the Eulerian description). As a result, it is equivalent to the system of characteristic ordinary differential equations (the Lagrangian description) describing particle dynamics:

$$
\frac{d}{dt} r(t) = v + v(r(t), t), \quad r(0) = r_0,
$$

(7.4)

$$
\frac{d}{dt} v(t) = -\lambda [v(t) - u(r(t), t)], \quad v(0) = v_0(r_0).
$$

These equations are the standard Newton equations for the dynamics of a particle with linear friction described by the Stokes force

$$
F(t) = -\lambda v(r(t), t)
$$

under the action of random force $f(t) = \lambda u(r(t), t)$ generated by the hydrodynamic flow.