A Method to Triangulate a Set of Points in the Plane

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Abstract. Given a set of points $S$ in the plane, we propose a triangulation process to construct a triangulation for the set $S$. Triangulating scattered point-sets is a very important problem in computational geometry; hence the importance to develop new efficient algorithms and models to triangulate point-sets. This process presents two well-distinguished phases; Phase 1 begins with the construction of an auxiliary triangular grid containing all the points. This auxiliary mesh will help us in the triangulation process, since we take this mesh as a reference to determine the points that may be joined by edges to construct triangles that will constitute the triangulation. Phase 2 takes as a starting point the auxiliary triangular mesh obtained in Phase 1. The objective now is to determine which of the vertices of the initial set $S$ must be joined to form the triangles that will constitute the triangulation of the points. Some examples are shown in detail to understand the behaviour of the triangulation process.

Keywords: Triangulation, grid, lattice, Delaunay triangulation, mesh.

1 Introduction

We can say, in a simple way, that a triangulation is a decomposition into triangles (see [7,13,16]). We are interested in triangulations of point sets in the Euclidean plane, where the input is a set of points in the plane, denoted by $S$, and a triangulation is defined as a subdivision of the convex hull of $S$ into triangles whose vertices are the points in $S$.

It is a well-known fact that $n$ points in the plane can have many different triangulations. Three important fields in which triangulations are frequently used are finite-element methods, terrain modeling and social science research. In the first case, triangulations are used to subdivide a complex domain by creating a mesh of simple elements (triangles), over which a system of differential equations

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can be solved more easily (see [9,10]). In the second case, the points represent sampled from a terrain, and the triangulation provides a bivariate interpolating surface, providing an elevation model of the terrain (see [5,17]). In the third case the concept of triangulation refers to a process by which a researcher wants to verify a finding by showing that independent measures of it agree with or, at least, do not contradict it ([14,15]).

In all these cases, the shapes of the triangles can have serious consequences on the result. For example, in finite-element methods, the aspect ratio of the triangles is particularly important, since elements of large aspect ratio can lead to poorly conditioned systems (see [2]).

In most applications, the need for well-shaped triangulations is usually addressed by using the Delaunay triangulation (see [3,4,6,8,11,12]). The Delaunay triangulation of a point set \( S \) is defined as a triangulation in which the vertices are the points in \( S \) and the circumcircle of each triangle (that is, the circle defined by the three vertices of each triangle) contains no other point from \( S \). The Delaunay triangulation has many known geometric properties and there exists an extensive bibliography with several efficient and relatively simple algorithms to compute it. Intuitively, we can say that its triangles are considered well shaped, that is, they maximize the minimum angle among all triangle angles, which implies that its angles are, in a sense, as large as possible.

2 Generation of a Triangular Mesh from a Set of Points

The primary objective that we set is the generation of a flat triangular mesh from a set of points \( S \) in the plane. That mesh, consisting of vertices and edges forming triangles, must have two main characteristics: may contain a number of points (vertices) greater than the initial set and its convex hull must contain all the points of the initial set.

In this process, we begin constructing a lattice graph; more exactly, we will draw a square grid graph and, afterwards, we will create from this grid a triangular grid graph that will be used as a starting point in the triangulation process. For the construction of this mesh we start from the cloud points. Let \( S \) be a set of \( n \) points in the plane \( S = \{P_1, P_2, \ldots, P_n\} \), whose coordinates are given by \( P_i = (x_i, y_i) \), for \( i = 1, 2, \ldots, n \).

2.1 Construction of a Square Grid from the Set of Points

The first step is to construct a square grid graph containing \( l \) vertices or nodes, such that \( l \approx 2n \); that is, the number of vertices of the new mesh should be, approximately, twice the number of points of \( S \). To carry out this task we are going to construct a lattice grid in a process that can be summarized as follows:

- We start computing the maximum and minimum values of the coordinates \( (x_i, y_i) \), for the points \( P_i \), for \( i = 1, 2, \ldots, n \). Using these parameters a rectangle must be created with all the points in its interior.