Modified Quaternion Newton Methods

Fernando Miranda and M. Irene Falcão

Departamento de Matemática e Aplicações and Centro de Matemática*,
Universidade do Minho, Portugal
{fmiranda,mif}@math.uminho.pt
http://www.math.uminho.pt

Abstract. We revisit the quaternion Newton method for computing roots of a class of quaternion valued functions and propose modified algorithms for finding *multiple* roots of simple polynomials. We illustrate the performance of these new methods by presenting several numerical experiments.

Keywords: Quaternion Analysis, Newton methods.

1 Introduction

In this work we concentrate on the problem of finding roots of special quaternion polynomials of the form

$$p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,\ a_i \in \mathbb{H},\ i = 0, \ldots, n,\ a_n \neq 0. \quad (1)$$

Since the work of Niven [18], several authors gave contributions to the problem of finding roots of quaternion valued polynomials (see e.g. [3,4,11,14,16,21,23]), by following different approaches and with different motivations.

Recently a quaternion version of the well known Newton method for finding roots of a class of quaternion functions was proposed in [5]. This work was motivated by [13], where the authors formally adapted, for the first time, Newton method for finding roots of quaternions, i.e. for solving quaternion equations of the form $x^n + a_0 = 0$.

Due to the non-commutativity of quaternion multiplication, the use of root-finding methods involving quaternion iterative functions requires close attention. In particular in the framework of Newton-like methods, left and right quaternion versions have to be considered.

The results of [5] are based on the equivalence between the classical multivariate Newton method and a quaternion version derived by the use of the so-called radial derivative.

In this work we give new insights on the quaternion Newton method, by making the link, under certain conditions, to the complex approach. Quaternion versions of well known variants of the classical Newton method for multiple roots are also derived.

* The research was partially supported by the Research Centre of Mathematics of the University of Minho with the Portuguese Funds from the "Fundação para a Ciência e a Tecnologia", through the Project PEstOE/MAT/UI0013/2014.

B. Murgante et al. (Eds.): ICCSA 2014, Part I, LNCS 8579, pp. 146–161, 2014.

2 Quaternion Analysis Toolbox

Let $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^4$ with a product given according to the multiplication rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}.$$

This non-commutative product generates the well known algebra of real quaternions $\mathbb{H}$. The real vector space $\mathbb{R}^4$ will be embedded in $\mathbb{H}$ by identifying the element $\boldsymbol{x} = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ with the element $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbb{H}$. Thus, throughout this paper, we will not distinguish an element in $\mathbb{R}^4$ and the corresponding quaternion in $\mathbb{H}$, unless we need to stress the context.

The conjugate of x is defined as

$$\bar{x} = x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k}$$

and instead of the real and the imaginary parts we will distinguish between the scalar part of x

$$\operatorname{Sc} x := x_0 = \tfrac{1}{2}(x + \bar{x})$$

and the vector part of x

$$\operatorname{Vec} x = \underline{x} := x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} = \tfrac{1}{2}(x - \bar{x}).$$

When $\operatorname{Sc} x = 0$, x is called a pure quaternion. The norm $|x|$ of x is defined by

$$|x|^2 = x\bar{x} = \bar{x}x = x_0^2 + x_1^2 + x_2^2 + x_3^2$$

and it immediately follows that each non-zero $x \in \mathbb{H}$ has an inverse given by

$$x^{-1} = \frac{\bar{x}}{|x|^2}.$$

Quaternions x such that $|x| = 1$ are called unit quaternions. Observe that any arbitrary non-real quaternion x can be written as

$$x = x_0 + \underline{x} = x_0 + \omega(\underline{x})|\underline{x}|\,, \tag{2}$$

where $\omega(\underline{x})$ is the unit quaternion

$$\omega(\underline{x}) = \frac{\underline{x}}{|\underline{x}|}\,,$$

very much like a complex number is written in the form $a + ib$. Moreover, since $\omega(\underline{x})^2 = -1$, one can argue that $\omega(\underline{x})$ behaves like the imaginary unit. In what follows we use the convention $\omega(\underline{x}) := 0$, for real quaternions x. Now, if x and y are quaternions such that $\omega(\underline{x}) = \omega(\underline{y}) =: \boldsymbol{\omega}$, i.e. if $x = a + \boldsymbol{\omega} b$ and $y = c + \boldsymbol{\omega} d$, then all the algebraic operations can be computed as if x and y were complex numbers, in particular,

$$xy = yx = ac - bd + \boldsymbol{\omega}(ad + bc).$$