Chapter 7
Minimum Energy Control
of Electrical Circuits

The minimum energy control problem for standard linear systems was formulated and solved by J. Klamka in [117, 118, 119, 120, 121, 122, 123]. The Klamka’s method has been extended to new classes of linear systems [12, 13, 71, 82, 83, 98, 84, 85, 86, 101, 94, 96, 97, 99, 100, 124].

7.1 Minimum Energy Control of Positive Standard Electrical Circuits

Consider the positive electrical circuit described by the equations (4.1) with diagonal $A \in M_n$ and monomial $B \in R_{+}^{n \times n}$. If the system is reachable in time $t \in [0, t_f]$, then usually there exists many different inputs $u(t) \in R_{+}^n$ that steers the state of the system from $x_0 = 0$ to $x_f \in R_{+}^n$. Among these inputs we are looking for an input that minimizes the performance index

$$I(u) = \int_0^{t_f} u^T(\tau)Q u(\tau) d\tau,$$

(7.1)

where $Q \in R_{+}^{n \times n}$ is a symmetric positive defined matrix and $Q^{-1} \in R_{+}^{n \times n}$.

The minimum energy problem for the positive electrical circuit (4.1) can be stated as follows [101]. Given the reachable matrices $A \in M_n$, $B \in R_{+}^{n \times n}$ and $Q \in R_{+}^{n \times n}$ of the performance index (7.1), $x_f \in R_{+}^n$ and $t_f > 0$, find an input $u(t) \in R_{+}^n$ for $t \in [0, t_f]$ that steers the state vector of the system from $x_0 = 0$ to $x_f \in R_{+}^n$ and minimizes the performance index (7.1).

To solve the problem we define the matrix

$$W = W(t_f, Q) = \int_0^{t_f} e^{A(t_f-\tau)} B Q^{-1} B^T e^{A^T(t_f-\tau)} d\tau.$$

(7.2)
From (7.2) and Theorem 5.7 it follows that the matrix (7.2) is monomial if and only if the positive electrical circuit (4.1) is reachable in time \([0, t_f]\).

In this case we may define the input

\[
\hat{u}(t) = Q^{-1}B^Te^{A^T(t_f-t)}W^{-1}x_f \quad \text{for} \quad t \in [0, t_f].
\]  

(7.3)

Note that the input (7.3) satisfies the condition \(u(t) \in \mathbb{R}^n_+\) for \(t \in [0, t_f]\) if\n
\[
Q^{-1} \in \mathbb{R}^{n \times n}_+ \quad \text{and} \quad W^{-1}x_f \in \mathbb{R}^n_+.
\]  

(7.4)

**Theorem 7.1.** Let the positive electrical circuit (4.1) be reachable in time \([0, t_f]\) and let \(\bar{u}(t) \in \mathbb{R}^n_+\) for \(t \in [0, t_f]\) be an input that steers the state of the positive electrical circuit (4.1) from \(x_0 = 0\) to \(x_f \in \mathbb{R}^n_+\). Then the input (7.3) also steers the state of the system from \(x_0 = 0\) to \(x_f \in \mathbb{R}^n_+\) and minimizes the performance index (7.1), i.e. \(I(\hat{u}) \leq I(\bar{u})\).

The minimal value of the performance index (7.1) is equal to

\[
I(\hat{u}) = x_f^TW^{-1}x_f.
\]  

(7.5)

**Proof.** If the conditions (7.4) are met, then the input (7.3) is well defined and \(\hat{u} \in \mathbb{R}^n_+\) for \(t \in [0, t_f]\). We shall show that the input steers the state of the system from \(x_0 = 0\) to \(x_f \in \mathbb{R}^n_+\).

Substitution of (7.3) into (5.24) for \(t = t_f\) and \(x_0 = 0\) yields

\[
x(t_f) = \int_0^{t_f} e^{A(t_f-\tau)}B\hat{u}(\tau)d\tau = \int_0^{t_f} e^{A(t_f-\tau)}BQ^{-1}B^Te^{A^T(t_f-\tau)}d\tau W^{-1}x_f = x_f,
\]

since (7.2) holds.

By the assumption, the inputs \(\bar{u}(t)\) and \(\hat{u}(t), \ t \in [0, t_f]\) steers the state of the system from \(x_0 = 0\) to \(x_f \in \mathbb{R}^n_+\). Hence

\[
x_f = \int_0^{t_f} e^{A(t_f-\tau)}B\bar{u}(\tau)d\tau = \int_0^{t_f} e^{A(t_f-\tau)}B\hat{u}(\tau)d\tau
\]  

(7.6a)

or

\[
\int_0^{t_f} e^{A(t_f-\tau)}B [\bar{u}(\tau) - \hat{u}(\tau)]d\tau = 0.
\]  

(7.6b)

By transposition of (7.6b) and postmultiplication by \(W^{-1}x_f\) we obtain

\[
\int_0^{t_f} [\bar{u}(\tau) - \hat{u}(\tau)]^T B^Te^{A^T(t_f-\tau)}d\tau W^{-1}x_f = 0.
\]  

(7.7)