The Fuzzy Description Logic $\mathbf{G-FL}_0$
with Greatest Fixed-Point Semantics*

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Abstract. We study the fuzzy extension of the Description Logic $\mathbf{FL}_0$ with semantics based on the Gödel t-norm. We show that subsumption w.r.t. a finite set of primitive definitions, using greatest fixed-point semantics, can be characterized by a relation on weighted automata. We use this result to provide tight complexity bounds for reasoning in this logic, showing that it is PSPACE-complete. If the definitions do not contain cycles, subsumption becomes co-NP-complete.

1 Introduction

Description logics (DLs) are used to describe the knowledge of an application domain in a formally well-defined manner [3]. The basic building blocks are concepts that intuitively describe a set of elements of the domain, and roles, which model binary relations over the domain. The expressivity of DLs is given by a set of constructors that are used to build complex concepts from so-called concept names, and is usually chosen to end up in decidable fragments of first-order predicate logic.

Knowledge about domain-specific terminology can be expressed by different kinds of axioms. For example, the concept definition

Father $\equiv$ Human $\sqcap$ Male $\sqcap$ $\exists$ hasChild. $\top$

is used to determine the extension of the concept name Father in terms of other concept names (Human, Male) and roles (hasChild). In contrast, a primitive concept definition like

Human $\sqsubseteq$ Mammal $\sqcap$ Biped

only bounds the interpretation of a concept name from above. Sometimes, one restricts (primitive) definitions to be acyclic, which means that the definition of a concept name cannot use itself (directly or indirectly via other definitions). In general concept inclusions (GCIs) such as

\[ \forall \text{hasParent.Human} \sqsubseteq \text{Human} \]

* Partially supported by the DFG under grant BA 1122/17-1, in the research training group 1763 (QuantLA), and the Cluster of Excellence ‘Center for Advancing Electronics Dresden’.

one can relate arbitrary complex expressions. These axioms are collected into so-called TBoxes, which can be either acyclic (containing acyclic definitions), cyclic (containing possibly cyclic definitions), or general (containing GCI). To interpret cyclic TBoxes, several competing semantics have been proposed [19].

Different DLs vary in the choice of constructors allowed for building complex concepts. For example, the small DL EL uses the constructors top (⊤), conjunction (⊓), and existential restriction (∃r.C for a role r and a concept C). We consider here mainly FL₀, which has top, conjunctions, and value restrictions (∀r.C). The DL ALC combines all the above constructors with negation (¬C).

Fuzzy description logics have been introduced as extensions of classical DLs capable of representing and reasoning with vague or imprecise knowledge. The main idea behind these logics is to allow for a set of truth degrees, beyond the standard true and false; usually, the real interval [0, 1] is considered. In this way, one can allow fuzzy concepts like Tall to assign an arbitrary degree of tallness to each individual, instead of simply classifying them into tall and not tall. Based on Mathematical Fuzzy Logic [13], a so-called t-norm defines the interpretation of conjunctions, and determines the semantics of the other constructors as well. The three main continuous t-norms are Gödel (G), Łukasiewicz (Ł), and Product (Π). The Zadeh semantics is another popular choice that is based on fuzzy set theory [25].

The area of fuzzy DLs recently experienced a shift, when it was shown that reasoning with GCI easily becomes undecidable [4,7,9]. To guarantee decidability in fuzzy DLs, one can (i) restrict the semantics to consider finitely many truth degrees [8]; (ii) allow only acyclic or unfoldable TBoxes [5,22]; or (iii) restrict to Zadeh or Gödel semantics [6,17,20,21].

In the cases where the Gödel t-norm is used, the complexity of reasoning is typically the same as for its classical version, as shown for subsumption w.r.t. GCI in G-EL, which is polynomial [17,20], and G-ALC, ExpTime-complete [6]. This latter result implies that subsumption in G-FL₀ with general TBoxes is also ExpTime-complete since it is ExpTime-hard already in classical FL₀ [2]. On the other hand, if TBoxes are restricted to contain only (cyclic) definitions, then deciding subsumption in classical FL₀ under the greatest fixed-point semantics is known to be PSpace-complete [1]. For acyclic TBoxes, the complexity reduces to co-NP-complete [18]. In this paper, we analyze reasoning in the Gödel extension of this logic.

Consider the cyclic definition of a tall person with only tall offspring (Toto):

\[ \text{Toto} \sqsubseteq \text{Person} \sqcap \text{Tall} \sqcap \forall \text{hasChild}.\text{Toto} \]

Choosing greatest fixed-point semantics is very natural in this setting, as it requires to always assign the largest possible degree for an individual to belong to Toto. Otherwise, Toto could simply assign degree 0 to all individuals, which is clearly not the intended meaning.

We show that the PS脢ACE-upper bound for reasoning in the classical case also applies to this fuzzy DL. To prove this, we characterize the greatest fixed-point semantics of G-FL₀ by means of [0, 1]-weighted automata. We then show that