Decomposition of Some Jacobian Varieties of Dimension 3

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Abstract. We study degree 2 and 4 elliptic subcovers of hyperelliptic curves of genus 3 defined over \( \mathbb{C} \). The family of genus 3 hyperelliptic curves which have a degree 2 cover to an elliptic curve \( E \) and degree 4 covers to elliptic curves \( E_1 \) and \( E_2 \) is a 2-dimensional subvariety of the hyperelliptic moduli \( \mathcal{H}_3 \). We determine this subvariety explicitly. For any given moduli point \( p \in \mathcal{H}_3 \) we determine explicitly if the corresponding genus 3 curve \( \mathcal{X} \) belongs or not to such family. When it does, we can determine elliptic subcovers \( E, E_1, \) and \( E_2 \) in terms of the absolute invariants \( t_1, \ldots, t_6 \) as in [12]. This variety provides a new family of hyperelliptic curves of genus 3 for which the Jacobians completely split.

The sublocus of such family when \( E_1 \cong E_2 \) is a 1-dimensional variety which we determine explicitly. We can also determine \( \mathcal{X} \) and \( E \) starting from the \( j \)-invariant of \( E_1 \).

1 Introduction

There are some problems in classical mathematics which can be solved only through symbolic computational methods. The problem in which this work is focused lies within this category. Whether methods in artificial intelligence, machine learning etc can be improved to generalize such computational methods remains to be seen.

Let \( \mathcal{M}_g \) denote the moduli space of genus \( g \geq 2 \) algebraic curves defined over an algebraically closed field \( k \) and \( \mathcal{H}_g \) the hyperelliptic submoduli in \( \mathcal{M}_g \). The sublocus of genus \( g \) hyperelliptic curves with an elliptic involution is a \( g \)-dimensional subvariety of \( \mathcal{H}_g \). For \( g = 2 \) this space is denoted by \( \mathcal{L}_2 \) and studied in [11] and for \( g = 3 \) is denoted by \( \mathcal{S}_2 \) and is computed and discussed in detail in [4]. In both cases, a birational parametrization of these spaces is found via dihedral invariants which are introduced by the second author and generalized for any genus \( g \geq 2 \) in [4]. We denote the parameters for \( \mathcal{L}_2 \) by \( u, v \) and for \( \mathcal{S}_2 \) by \( s_2, s_3, s_4 \) as in respective papers. Hence, for the case \( g = 3 \) there is a birational map \( \phi : \mathcal{S}_2 \rightarrow \mathcal{H}_3 \) such that \( \phi : (s_2, s_3, s_4) = (t_1, \ldots, t_6) \), where \( t_1, \ldots, t_6 \) are the absolute invariants as defined in [12].

The dihedral invariants \( s_2, s_3, s_4 \) provide a birational parametrization of the locus \( \mathcal{S}_2 \). Hence, a generic curve in \( \mathcal{S}_2 \) is uniquely determined by the corresponding triple \( (s_2, s_3, s_4) \). Let \( \mathcal{X} \) be a curve in the locus \( \mathcal{S}_2 \). Then there is a degree 2 map \( f_1 : \mathcal{X} \rightarrow E \) for some elliptic curve \( E \). Thus, the Jacobian of \( \mathcal{X} \) splits as \( \text{Jac}(\mathcal{X}) \cong E \times A \), where \( A \) is a genus 2 Jacobian. Hence, there is a map
f_2 : X \rightarrow C for some genus 2 curve C. The equations of X, E, and C are given in Thm. [2]. For any fixed curve X ∈ S_2, the subcovers E and C are uniquely determined in terms of the invariants s_2, s_3, s_4.

In section three we give the splitting of the Jacobians for all genus 3 algebraic curves, when this splitting is induced by automorphisms. The proof requires the Poincare duality and some basic group theory.

In this paper, we are mostly interested in the case when the Jacobian of the genus two curve C also splits. The Jacobian of C can split as an (n,n)-structure; see [8]. The loci of such genus 2 curves with (3,3)-split or (5,5)-split have been studied respectively in [7, 9]. For n = 4 the reader can check [5]. We focus on the case when the Jacobian of C is (2,2)-split, which corresponds to the case when the Klein 4-group V_4 ↪ Aut (C). Hence, Jac X splits completely as a product of three elliptic curves. We say that Jac X is (2,2,4)-split.

Let the locus of genus 3 hyperelliptic curves whose Jacobian is (2,4,4)-split be denoted by T. Then, there is a rational map ψ : T → L_2 such that ψ(s_2, s_3, s_4) = (u, v), which has degree 70 and can be explicitly computed, even though the rational expressions of u and v in terms of s_2, s_3, s_4 are quite large.

There are three components of T which we denote them by T_i, i = 1, 2, 3. Two of these components are well known and correspond to the cases when V_4 is embedded in the reduced automorphism group of X. These cases correspond to the singular locus of S_2 and are precisely the locus det(Jac (φ)) = 0. This happens for all genus g ≥ 2 as noted in [11]. The third component T_3 is more interesting to us. It doesn’t seem to have any group theoretic reason for this component to be there in the first place. We find the equation of this component in terms of the s_2, s_3, s_4 invariants. It is an equation F_1(s_2, s_3, s_4) = 0 as in Eq. (7). In this locus, the elliptic subfields of the genus two field k(C) can be determined explicitly.

The main goal of this paper is to determine explicitly the family T_3 of genus 3 curves and relations among its elliptic subcovers. We have the maps T_3 ↪ L_2 ↪ k^2, such that ψ_0(ψ(s_2, s_3, s_4)) = (j_1, j_2), where s_2, s_3, s_4 satisfy Eq. (7) and u, v are given explicitly by Eq. (6) and Thm. (3) in [11]. The degree deg ψ_0 = 2 and deg ψ = 70.

Since T_3 is a subvariety of H_3 it would be desirable to express its equation in terms of a coordinate in H_3. One can use the absolute invariants of the genus 3 hyperelliptic curves t_1, ..., t_6 as defined in [12] and the expressions of s_2, s_3, s_4 in terms of these invariants as computed in [4].

Further, we focus our attention to the sublocus V of L_2 such that the genus 2 field k(C) has isomorphic elliptic subfields. Such locus was discovered in [11] and it is somewhat surprising. It does not rise from a family of genus two curves with a fixed automorphism group as other families, see [11] for details. Using this sublocus of M_2 we discover a rather unusual embedding M_1 ↪ M_2 as noted in [11]. Let T ⊂ T_3 ⊂ H_3 be the subvariety of T_3 obtained by adding the condition