Chapter 7

Positive Definite Functions and Kernels, and Reproducing Kernel Hilbert Spaces

... afin de pouvoir mieux mettre en lumière quelques idées simples, fondamentales dans l'Analyse harmonique non commutative. Une de ces notions, celle de fonction de type positif, apparaît comme fortuite et quelque peu artificielle tant qu'on ne s'est pas rendu compte qu'elle est la contrepartie, pour les représentations linéaires des groupes, de la notion de représentation monogène.

Jean Dieudonné, [112, p. 2]

Positive definite kernels (and the associated reproducing kernel Hilbert spaces) play an important role in various fields in mathematics, and Dieudonné’s judgment is very harsh, and somewhat unjustified. Besides representation theory and harmonic analysis, they appear in function theory (the kernel function associated with a domain), in stochastic processes (every positive definite function is a correlation function, and vice versa; see Michel Loève’s book [223]), in infinite-dimensional analysis, in learning theory (see for instance [276, 232, 291, 181]), and in linear system theory (positivity translates into dissipativity of some underlying linear system), to name a few. In this chapter we present exercises which reflect some of this diversity. We refer to the books [267, 268] of Saitoh and to the papers [179] by Hille and [302, 301] by Szafraniec for background information and applications.
7.1 Positive definite kernels

We first recall the definition of a positive definite kernel.

**Definition 7.1.1.** Let \( \Omega \) be a set. The function \( K(z, w) \) from \( \Omega \times \Omega \) into \( \mathbb{C} \) (one uses usually the term kernel in this framework) is called a positive definite kernel if the following condition holds: For every \( N \in \mathbb{N} \), every choice of \( w_1, \ldots, w_N \in \Omega \), and every choice of \( c_1, \ldots, c_N \in \mathbb{C} \),

\[
\sum_{j,k=1}^{N} \overline{c_j} K(w_j, w_k) c_k \geq 0. \tag{7.1.1}
\]

We already met a number of examples earlier in the book. In particular:

**Question 7.1.2.** Show that the kernel (1.6.14) is positive definite in the open unit disk.

**Hint:** Use Exercise 2.1.36.

Condition (7.1.1) is equivalent to saying that the \( N \times N \) matrix with \((j, k)\) entry \( K(w_j, w_k) \) is positive. More generally, there is the following fundamental notion, due to Kreĭn:

**Definition 7.1.3.** Let \( \Omega \) be a set. The function \( K(z, w) \) from \( \Omega \times \Omega \) into \( \mathbb{C} \) has a finite number of negative squares, say \( \kappa \), if the following condition holds: It is Hermitian,

\[
K(z, w) = \overline{K(w, z)}, \quad \forall z, w \in \Omega,
\]

and for every \( N \in \mathbb{N} \), every choice of \( w_1, \ldots, w_N \in \Omega \), the \( N \times N \) matrix with \((j, k)\) entry \( K(w_j, w_k) \) has at most \( \kappa \) strictly negative eigenvalues, and exactly \( \kappa \) strictly negative eigenvalues for some choice of \( N \) and \( w_1, \ldots, w_N \).

Assume now that \( \Omega \) has a group structure with operation denoted by \( \circ \) and where the inverse of \( a \in \Omega \) is denoted by \( a^{-1} \). When \( \Omega \) is Abelian, the function \( f \) from \( \Omega \) into \( \mathbb{C} \) is called a positive definite function if the kernel \( f(z \circ w^{-1}) \) is positive definite in \( \Omega \). For instance:

**Definition 7.1.4.** The function \( f(t) \) of the real variable \( t \) is called positive definite if the associated kernel \( f(t - s) \) is positive definite in \( \mathbb{R} \) in the sense of Definition 7.1.1.

When \( \Omega \) is not Abelian, one can consider this latter kernel, but also the kernel \( f(w^{-1} \circ z) \). The two kernels need not be simultaneously positive definite. When \( \Omega \) is a finite-dimensional real vector space, the function \( f \) has a special structure. Setting \( \Omega = \mathbb{R}^N \), Bochner’s theorem states that, provided \( f \) is continuous, there

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1Note that the terminology *positive definite* is a bit misleading since the inequality in (7.1.1) is not strict.