3

First order nonlinear differential equations

The main focus of this chapter is on learning how to solve certain classes of nonlinear differential equations of first order.

3.1 Separable equations

An equation of the form

\[ x' = h(t)g(x) \]  

(3.1)

is called a separable equation. Let us assume that \( h(t) \) is continuous with \( h(t) \neq 0 \) and \( g(x) \) is continuously differentiable in the domain being considered, so that the local existence and uniqueness Theorem 2.2.1 of Chapter 2 applies.

If \( x = k \) is any zero of \( g \), \( g(k) = 0 \), then \( x(t) \equiv k \) is a constant solution of (3.1). On the other hand, if \( x(t) = k \) is a constant solution, then we would have \( 0 = h(t)g(k), \ t \in \mathbb{R} \), and hence \( g(k) = 0 \) since \( h(t) \neq 0 \). Therefore, \( x(t) = k \) is a constant solution (or equilibrium solution) if and only if \( g(k) = 0 \). There are no other constant solutions. All the non-constant solutions are separated by the straight lines \( x = k \). Hence if \( x(t) \) is a non-constant solution then \( g(x(t)) \neq 0 \) for any \( t \), and we can divide

\[ x' = h(t)g(x) \]

by \( g(x) \) yielding

\[ \frac{x'(t)}{g(x(t))} = h(t). \]

We integrate both sides with respect to \( t \) and obtain

\[ \int \frac{x'(t)}{g(x(t))} dt = \int h(t) dt. \]

Since \( x' = \frac{dx}{dt} \), we have

\[ \int \frac{dx}{g(x)} = \int h(t) dt + c. \]  

(3.2)
We wish to point out that while it is very easy to express solutions of a separable equation implicitly in terms of integrals, it may be difficult or even impossible to perform the actual integration in terms of simple and familiar functions. In such cases, one can carry out a qualitative analysis to get some information about the behavior of solutions, see for example Section 2.3 in the previous chapter. Otherwise, if needed, one could use numerical methods or computer software to obtain approximate solutions to a reasonable or needed degree of accuracy.

If we want to solve the initial value problem

\[
\begin{cases}
x' &= h(t)g(x) \\
x(t_0) &= x_0
\end{cases}
\]

we simply substitute the initial value \(x(t_0) = x_0\) into (3.2) and solve for \(c\). Note that this equation has a unique solution, according to Theorem 2.2.1 of Chapter 2.

Essentially, the idea behind solving separable equations is to separate the variables and then integrate.

**Example 3.1.1.** (i) Consider the equation \(x' = h(t)x\). We notice that this is a linear homogeneous first order equation, and we learned in Chapter 1 how to solve such equations. But this is also a separable equation and we can solve it by the method described above. Separating the variables and then integrating, we obtain \(\int \frac{dx}{x} = \int h(t)dt + c\), which yields \(\ln |x| = \int h(t)dt + c\). Thus, letting \(c_1 = \pm e^c\), we obtain the general solution

\[x(t) = c_1 e^{\int h(t)dt}\]

in accordance with the result stated in Theorem 1.4.2 of Chapter 1.

(ii) Solve \(x' = \frac{t^2}{1+3x^2}\).

There are no constant solutions. Separating the variables and integrating, we have \(\int (1 + 3x^2)dx = \int t^2dt + c\) and hence

\[x + x^3 = \frac{t^3}{3} + c\]

which defines the solutions implicitly. Moreover, since the function \(\Phi(x) = x + x^3\) is increasing and its image is all of \(\mathbb{R}\), it has an (increasing) inverse \(\varphi\) defined on all of \(\mathbb{R}\). Thus \(\Phi(x) = \frac{t^3}{3} + c\) yields \(x(t) = \varphi(\frac{t^3}{3} + c)\). Note that the solutions are defined globally on \(\mathbb{R}\). The reader might check that this also follows from the Global Existence Theorem 2.2.10 of Chapter 2.

(iii) Find the solution of the initial value problem \(x' = 2tx^3\), \(x(0) = 1\).

The only constant solution is \(x \equiv 0\). Therefore if \(x(t)\) is a solution such that \(x(0) = 1\), then, by uniqueness, \(x(t)\) cannot assume the value 0 anywhere. Since \(x(0) = 1 > 0\), we infer that the solution is always positive. Using (3.2) we find

\[
\int \frac{dx}{x^3} = \int 2t\,dt + c.
\]