LECTURES ON SUPERSINGULAR K3 SURFACES AND THE CRYSTALLINE TORELLI THEOREM

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Abstract. We survey crystalline cohomology, crystals, and formal group laws with an emphasis on geometry. We apply these concepts to K3 surfaces, and especially to supersingular K3 surfaces. In particular, we discuss stratifications of the moduli space of polarized K3 surfaces in positive characteristic, Ogus’ crystalline Torelli theorem for supersingular K3 surfaces, the Tate conjecture, and the unirationality of K3 surfaces.

Introduction

In these notes, we cover the following topics

- algebraic de Rham cohomology, crystalline cohomology, and $F$-crystals,
- characteristic-$p$ aspects of K3 surfaces,
- Ogus’ crystalline Torelli theorem for supersingular K3 surfaces,
- formal group laws, and in particular, the formal Brauer group, and
- unirationality and supersingularity of K3 surfaces.

We assume familiarity with algebraic geometry, say, at the level of the textbooks of Hartshorne [Har77] and Griffiths–Harris [G-H78].

One aim of these notes is to convince the reader that crystals and crystalline cohomology are rather explicit objects, and that they are characteristic-$p$ versions of Hodge structures and de Rham cohomology, respectively. Oversimplifying and putting it a little bit sloppily, the crystalline cohomology of a smooth and proper variety in characteristic $p$ is the de Rham cohomology of a lift to characteristic zero. (Unfortunately, such lifts may not exist, and even if they do, they may not be unique - it was Grothendieck’s insight that something like crystalline cohomology exists nevertheless and that it is well-defined.) Just as complex de Rham cohomology comes with complex conjugation, crystalline cohomology comes with a Frobenius action, and this latter leads to the notion of

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a crystal. Therefore, period maps in characteristic \( p \) should take values in moduli spaces of crystals. For example, Ogus’ crystalline Torelli theorem states that moduli spaces of certain K3 surfaces, namely, supersingular K3 surfaces, can be entirely understood via a period map to a moduli space of suitably enriched crystals. Conversely, the classification of crystals arising from K3 surfaces gives rise to a stratification of the moduli space of K3 surfaces in characteristic \( p \); for example, the height stratification in terms of Newton polygons arises this way.

A second aim of these notes is to introduce formal group laws, and, following Artin and Mazur, to explain how they arise naturally from algebraic varieties. For us, the most important examples will be the formal Picard group and the formal Brauer group of a smooth and proper variety in characteristic \( p \). Whereas the former arises as formal completion of the Picard scheme along its origin, the latter does not have such a description, and is something new. Oversimplifying again, we have a sort of crystal associated to a formal group law, namely, its Cartier–Dieudonné module. For example, the Cartier–Dieudonné modules of the formal Picard group and the formal Brauer group give rise to subcrystals inside first and second crystalline cohomology of the variety in question. And despite their abstract appearance, these formal group laws do have a geometric interpretation: for example, for supersingular K3 surfaces, the formal Brauer group controls certain very special one-parameter deformations, moving torsors, which are characteristic-\( p \) versions of twistor space. These deformations are the key to the proof that supersingular K3 surfaces are unirational.

And finally, a third aim of these notes is to make some of the more abstract concepts more accessible, which is why we have put an emphasis on computing everything for K3 surfaces. Even for a reader who cares not so much for K3 surfaces, these notes may be interesting, since we show by example, how to perform computations with crystals and formal Brauer groups.

These notes are organized as follows:

Section 1 We start by discussing de Rham cohomology over the complex numbers, and then turn to algebraic de Rham cohomology. After a short detour to \( \ell \)-adic cohomology, we introduce the Witt ring \( W \), and survey crystalline cohomology.

Section 2 We define K3 surfaces, give examples, and discuss their position within the surface classification. Then, we compute their cohomological invariants, and end by introducing polarized moduli spaces.

Section 3 Crystalline cohomology takes it values in \( W \)-modules, where \( W \) denotes the Witt ring, and it comes with a Frobenius-action, which leads to the notion of an \( F \)-crystal. After discussing the Dieudonné–Manin classification of \( F \)-crystals up to isogeny, we introduce Hodge and Newton polygons of \( F \)-crystals.

Section 4 The \( F \)-crystal associated to the second crystalline cohomology group of a K3 surface comes with a quadratic form arising from Poincaré duality, which is captured in the notion of a K3 crystal. After discussing supersingular