Chapter 14

Some Useful Theorems

In this chapter we collect a number of results from real analysis, which are useful to solve the exercises. The results presented are along one main theme: How to interchange two operations in analysis (for instance order of integration in a double integral, integration of a function depending on a parameter and derivation with respect to this parameter,...). Most, if not all, of the results, can be proved by elementary methods, but are also special cases of general theorems from the theory of integration (such as the dominated convergence theorem, Fubini’s theorem,...). Some aspects of this theory are reviewed in Chapter 17. Finally, note that we consider complex-valued functions. The results are easily derived in the complex case from their real counterparts. In fact, they are sometimes still valid for functions and sequences with values in a Banach space or a Banach algebra, but a discussion of this latter point is far outside the framework of this book.

14.1 Differentiable functions of two real variables

We here recall the definition of a differentiable function of two real variables. The case of functions with domain and range inside Banach spaces is given in Section 16.1. See Definition 16.1.13.

Definition 14.1.1. A real-valued function $t(x, y)$ defined in a neighborhood of the point $(x_0, y_0) \in \mathbb{R}^2$ is said to be differentiable at $(x_0, y_0)$ if there exist real numbers $a$ and $b$ such that

$$
\lim_{x \to x_0, y \to y_0} \frac{t(x, y) - t(x_0, y_0) - a(x - x_0) - b(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.
$$

(14.1.1)

It is well known that a necessary (but in no way sufficient) condition for differentiability at the point $(x_0, y_0)$ is that $t$ has first-order partial derivatives at
this point. The numbers $a$ and $b$ are unique and equal to

$$a = \frac{\partial t}{\partial x}(x_0, y_0) \quad \text{and} \quad b = \frac{\partial t}{\partial y}(x_0, y_0)$$

Differentiability can be written in an equivalent way as follows: The function $t$ admits first-order partial derivatives at the point $(x_0, y_0)$ and there exists a function $E(x, y)$ such that

$$t(x, y) = t(x_0, y_0) + (x - x_0) \frac{\partial t}{\partial x}(x_0, y_0) + (y - y_0) \frac{\partial t}{\partial y}(x_0, y_0)$$

$$+ \sqrt{(x - x_0)^2 + (y - y_0)^2} E(x, y)$$

(14.1.2)

and

$$\lim_{x \to x_0, \; y \to y_0} E(x, y) = 0.$$

The function $E(x, y)$ is uniquely defined, and is equal to

$$E(x, y) = \frac{t(x, y) - t(x_0, y_0) - (x - x_0) \frac{\partial t}{\partial x}(x_0, y_0) - (y - y_0) \frac{\partial t}{\partial y}(x_0, y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}.$$  (14.1.3)

Condition (14.1.2) is often more convenient than (14.1.1) to work with.

The following classical counter-example shows that continuity of the function and existence of partial derivatives at a given point do not imply differentiability at that point.

**Example 14.1.2.** Let

$$t(x, y) = \begin{cases} 
\frac{xy}{\sqrt{x^2 + y^2}}, & \text{if } (x, y) \neq (0, 0), \\
0, & \text{if } (x, y) = (0, 0).
\end{cases}$$

Then, $t$ is continuous at the point $(0,0)$, but is not differentiable there.

**Discussion.** The continuity at the origin follows from the inequality

$$|t(x, y)| \leq \frac{\left(\frac{x^2 + y^2}{2}\right)}{\sqrt{x^2 + y^2}} = \frac{\sqrt{x^2 + y^2}}{2}, \quad (x, y) \neq (0, 0).$$

The partial derivatives at the origin exist and are equal to 0, as follows from

$$\frac{t(x,0) - t(0,0)}{x} \equiv 0 \quad \text{and} \quad \frac{t(0,y) - t(0,0)}{y} \equiv 0.$$