Chapter 2

Complex Numbers: Geometry

As is well known, the complex field can be identified with \( \mathbb{R}^2 \) via the map

\[ z = x + iy \mapsto (x, y). \]

An important new feature with respect to real analysis is the introduction of the point at infinity, which leads to the compactification of \( \mathbb{C} \). These various aspects, and some others, such as Moebius maps, are considered in this chapter.

2.1 Geometric interpretation

Exercise 2.1.1. Describe the polygon whose vertices are defined by the roots of order \( n \) of unity.

To have a good understanding of some forthcoming notions (for instance, limit at infinity, or the notion of pole of an analytic function), it is better to be able to leave the complex plane, and go one step further and add a point, called infinity, and denoted by the symbol \( \infty \) (without sign, in opposition to real analysis, where you have \( \pm \infty \)), in such a way that the extended complex plane \( \mathbb{C} \cup \{\infty\} \) is compact. The set

\[ \mathbb{C} \cup \{\infty\} \]

is called the extended complex plane. See Section 15.1 for a reminder of the notion of compactness. For the topological details of the construction, see Section 15.3. In the next exercise we discuss the geometric interpretation of the point at infinity, by identifying the extended complex plane with the Riemann sphere

\[ \mathbb{S}_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 ; x_1^2 + x_2^2 + x_3^2 = 1\}. \]
Exercise 2.1.2. For \((x_1, x_2, x_3) \in S_2 \setminus \{(0, 0, 1)\}\), define \(\varphi(x_1, x_2, x_3)\) to be the intersection of the line defined by the points \((0, 0, 1)\) and \((x_1, x_2, x_3)\) with the complex plane. Show that

\[
\varphi(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3},
\]

and that \(\varphi\) is a bijection between \(S_2 \setminus \{(0, 0, 1)\}\) and \(\mathbb{C}\), with inverse given by

\[
\varphi^{-1}(u + iv) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}\right).
\]

Setting \(z = u + iv\), (2.1.2) may be rewritten as

\[
\varphi^{-1}(z) = \left(\frac{z + \overline{z}}{|z|^2 + 1}, \frac{z - \overline{z}}{i(|z|^2 + 1)}, \frac{|z|^2 - 1}{|z|^2 + 1}\right).
\]

The map (2.1.1) is called the stereographic projection.

The geometrical interpretation of the point at infinity is as follows: The map \(\varphi\) is extended to the point \((0, 0, 1)\) by

\[
\varphi(0, 0, 1) = \infty,
\]

and going to \(\infty\) on the complex plane means going to \((0, 0, 1)\) on the Riemann sphere. More precisely, recall that, by definition, a sequence of complex numbers \((z_n)_{n \in \mathbb{N}}\) tends to infinity if

\[
\lim_{n \to \infty} |z_n| = +\infty,
\]

that is, if and only if

\[
\lim_{n \to \infty} \varphi^{-1}(z_n) = (0, 0, 1),
\]

where this last limit can be understood in two equivalent ways: The first, and simplest, is just to say that the limit is coordinate-wise in \(\mathbb{R}^3\). The second is to view \(S_2\) as a topological manifold, and see the limit in the corresponding topology. See also Exercise 15.1.5, where \(\varphi\) allows us to define a metric on the Riemann sphere, called the stereographic metric.

The intersection of \(S_2\) with a (non-tangent) plane is a circle. Note that the projection of a circle of the Riemann sphere on the plane will not be a circle in general. For instance the projection of the circle

\[
x_1 = x_3,
x_1^2 + x_2^2 + x_3^2 = 1
\]

onto the plane is the ellipse \(2x_1^2 + x_2^2 = 1\). But we have: