Quite often it is not possible to postulate an appropriate parametric form for the DGP under study. In such cases, semi- and nonparametric methods are called for. Certain of these methods introduced in Chapter 9 can be easily extended to the multivariate (vector) framework. Specifically, let $\mathbf{Y}_t = (Y_{1,t}, \ldots, Y_{m,t})'$ denote an $m$-dimensional process. We consider again the general nonlinear VAR($p$) model

$$Y_{\ell,t} = f_{\ell}(\mathbf{Y}_{t-1}, \ldots, \mathbf{Y}_{t-p}) + \varepsilon_{\ell,t}, \quad (\ell = 1, \ldots, m), \quad (12.1)$$

where $\varepsilon_t = (\varepsilon_{1,t}, \ldots, \varepsilon_{m,t})'$ is an $m$-dimensional i.i.d. variable with mean vector 0 and $m \times m$ covariance matrix $\Sigma_{\varepsilon}$, independent of $\mathbf{Y}_t$. In this chapter, we discuss various aspects related to data-driven estimation and forecasting methods, as well as to the detection of dependence structures and interrelationships in multivariate time series.

In Section 12.1, we start off by extending the theory of univariate kernel-based conditional quantile estimation to higher dimensions. In addition, we present a kernel-based forecasting method. Valuable as these methods can sometimes be, the increase in the dimensionality of the predictor space makes straightforward application of kernel-based methods impractical in practice unless both $m$ and $p$ are small and $T$ is large. As an alternative, constraining the functions $f_{\ell}(\cdot)$ in (12.1) in such a way that they still provide flexible representations of the unknown underlying functions yet do not suffer from excessive data requirements results is often a more useful approach. Of the semiparametric methods discussed in Chapter 9, (TS)MARS, k-NN, PPR and FCAR are most easily extended to the multivariate framework; see Section 12.2. In Section 12.3, we discuss vector frequency-domain Gaussianity and linearity test statistics.

In Section 12.4, we turn our attention to an exploratory nonparametric test statistic for lag identification in vector nonlinear time series which is a multivariate analogue to the mutual information coefficient $R(\cdot)$ given by (1.20). Finding appropriate lags for inclusion in a vector nonlinear time series model can be based...
on this test statistic and hence it can serve as an initial way to infer causal relationships. In Section 12.5, we then introduce three formal nonlinear causality test statistics. These tests are closely related to test statistics for high-dimensional serial independence, which we discussed earlier in Chapter 7.

Two appendices are added to the chapter. Appendix 12.A provides information about the numerical computation of multivariate conditional quantiles. Appendix 12.B discusses how to compute percentiles of the vector based analogue of the univariate test statistic \( \hat{R}_Y(\cdot) \) introduced in Section 1.3.3.

12.1 Nonparametric Methods

12.1.1 Conditional quantiles

Suppose that data are available in the form of a strictly stationary stochastic process \( \{(X_t, Y_t), t \in \mathbb{Z}\} \) with the same distribution as \((X, Y)\) taking values in \(\mathbb{R}^{mp} (p \geq 1, m \geq 2)\). Our aim is to generalize the univariate conditional quantile definition of Section 9.1.2 into a multivariate setting, i.e., \(m \geq 2\). First, we introduce some notation.

Let \( \| \cdot \|_{s,q} : \mathbb{R}^m \to \mathbb{R} \), be the application defined by

\[
\| z \|_{s,q} = \left\| \left( z_1, \ldots, z_m \right) \right\|_{s,q} = \left\| \frac{|z_1| + (2q - 1)z_1}{2}, \ldots, \frac{|z_m| + (2q - 1)z_m}{2} \right\|_s.
\]

Although \( \| \cdot \|_{s,q} \) is not a norm on \(\mathbb{R}^m\), it has properties similar to those of a norm; see Abdous and Theodorescu (1992). Below, we consider the Euclidean norm. Furthermore, for notational simplicity, we write \( \| \cdot \|_q \) for \( \| \cdot \|_{2,q} \), and \( \| \cdot \| \) for \( \| \cdot \|_2 \).

For a fixed \( x \in \mathbb{R}^p \), we define a vector function of \( \theta (\theta \in \mathbb{R}^m) \) by

\[
\varphi(\theta, x) = \mathbb{E}( \| Y - \theta \|_q - \| Y \|_q | X = x ) = \int_{\mathbb{R}^m} ( \| y - \theta \|_q - \| y \|_q ) Q(dy|x), \tag{12.2}
\]

where \( Q(\cdot|x) \) is the conditional probability measure of \( Y_t \) given \( X_t = x \). Because \( \| \theta \|_q < \| \theta \| \), we have \( |\varphi(\theta, x)| \leq \| \theta \| \) \( \forall \theta \in \mathbb{R}^m \). Thus, \( \varphi(\cdot, x) \) is well-defined. We shall call a \(q\)-conditional multivariate quantile, any point \( \theta_q(x) \) which assumes the infimum

\[
\varphi(\theta_q(x), x) = \inf_{\theta \in \mathbb{R}^m} \varphi(\theta, x). \tag{12.3}
\]

Unless \( Q(\cdot|x) \) is included into a straight line in \(\mathbb{R}^m\), it can be shown (Kemperman, 1987, Thm. 2.17) that \( \varphi(\theta, x) \) must be a strictly convex function of \( \theta \), assuming \( \| \cdot \|_q \) is a strictly convex norm (Appendix 3.A). This guarantees the existence and uniqueness of \( \theta_q(x) \). If the norm is not strictly convex, uniqueness of \( \varphi(\cdot, x) \) is not guaranteed; see, e.g., Oja (1983). Also, when \( \varphi(\cdot, x) \) is defined on an infinite-dimensional space, it may have no minimum (León and Massé, 1992).