Chapter 1

Monoidal categories and functors

The study of monoidal categories originated in the work of Jean Bénabou [Ben] and Saunders Mac Lane [ML1]. In this chapter, we review the basics of the theory of monoidal categories.

1.1 Categories and functors

1.1.1 Categories

A category \( \mathcal{C} \) consists of the following data:

- a class \( \text{Ob}(\mathcal{C}) \), whose elements are called objects of \( \mathcal{C} \);
- for any \( X, Y \in \text{Ob}(\mathcal{C}) \), a set \( \text{Hom}_\mathcal{C}(X, Y) \), whose elements are called morphisms from \( X \) to \( Y \) and represented by arrows \( X \to Y \);
- for any \( X, Y, Z \in \text{Ob}(\mathcal{C}) \), a map
  \[
  \circ : \text{Hom}_\mathcal{C}(Y, Z) \times \text{Hom}_\mathcal{C}(X, Y) \to \text{Hom}_\mathcal{C}(X, Z)
  \]
  called composition. The image of a pair \( (g, f) \) under this map is denoted \( g \circ f \) or just \( gf \);
- for every \( X \in \text{Ob}(\mathcal{C}) \), a morphism \( \text{id}_X \in \text{Hom}_\mathcal{C}(X, X) \), called the identity of \( X \).

It is required that the composition is associative and unitary in the following sense:

\[
(h \circ g) \circ f = h \circ (g \circ f) \quad \text{and} \quad f \circ \text{id}_X = f = \text{id}_Y \circ f
\]

for all morphisms \( f : X \to Y, g : Y \to Z, h : Z \to T \) with \( X, Y, Z, T \in \text{Ob}(\mathcal{C}) \).
Given a morphism $f : X \to Y$ in a category $\mathcal{C}$, the object $X$ is called the source and the object $Y$ the target of $f$. Two morphisms $g, f$ in $\mathcal{C}$ are composable if the source of $g$ coincides with the target of $f$. For $X \in \text{Ob}(\mathcal{C})$, the set $\text{Hom}_\mathcal{C}(X, X)$ is denoted by $\text{End}_\mathcal{C}(X)$, and its elements are called endomorphisms of $X$. The set $\text{End}_\mathcal{C}(X)$ is a monoid with product $gf = g \circ f$ for any $f, g \in \text{End}_\mathcal{C}(X)$ and unit $\text{id}_X$.

A morphism $f : X \to Y$ in $\mathcal{C}$ is an isomorphism if there exists a morphism $g : Y \to X$ in $\mathcal{C}$ such that $gf = \text{id}_X$ and $fg = \text{id}_Y$. Such a $g$ is uniquely determined by $f$, is called the inverse of $f$ and denoted $f^{-1}$. Two objects $X, Y$ of $\mathcal{C}$ are isomorphic if there exists an isomorphism $X \to Y$. Isomorphism of objects is an equivalence relation on $\text{Ob}(\mathcal{C})$ denoted $\simeq$.

The opposite of a category $\mathcal{C}$ is the category $\mathcal{C}^{\text{op}}$ defined by $\text{Ob}(\mathcal{C}^{\text{op}}) = \text{Ob}(\mathcal{C})$ and $\text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X)$ for all $X, Y \in \text{Ob}(\mathcal{C})$ with composition $\circ^{\text{op}}$ defined by $g \circ^{\text{op}} f = fg$.

A subcategory of a category $\mathcal{C}$ is a category $\mathcal{D}$ such that every object of $\mathcal{D}$ is an object of $\mathcal{C}$, for any $X, Y \in \text{Ob}(\mathcal{D})$, the set $\text{Hom}_\mathcal{D}(X, Y)$ is a subset of $\text{Hom}_\mathcal{C}(X, Y)$, the composition in $\mathcal{D}$ is the restriction of that in $\mathcal{C}$, and the identity morphisms in $\mathcal{D}$ are the same as in $\mathcal{C}$. A subcategory $\mathcal{D}$ of $\mathcal{C}$ is full if $\text{Hom}_\mathcal{D}(X, Y) = \text{Hom}_\mathcal{C}(X, Y)$ for all $X, Y \in \text{Ob}(\mathcal{D})$.

1.1.2 Example

Sets and maps between them form a category denoted $\text{Set}$. Finite sets and maps between them form a full subcategory of $\text{Set}$.

1.1.3 Example

Left modules over the ring $k$ and $k$-linear homomorphisms (with the usual composition) form a category denoted $\text{Mod}_k$.

1.1.4 Example

Given a set $G$, we define a category $G_k$ as follows. The objects of $G_k$ are elements of $G$. By definition, $\text{Hom}_{G_k}(g, g) = k$ for all $g \in G$ and $\text{Hom}_{G_k}(g, h) = \{0\} \subset k$ for any distinct $g, h \in G$. The composition of morphisms in $G_k$ is induced by multiplication in $k$. The identity of an object $g \in G$ is $\text{id}_g = 1_k$.

1.1.5 Functors and natural transformations

Functors are morphisms of categories and natural transformations are morphisms of functors. More precisely, a functor $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ assigns to each object $X$ of $\mathcal{C}$ an object $F(X)$ of $\mathcal{D}$ and to each morphism $f : X \to Y$ in $\mathcal{C}$ a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$ so that

$$F(gf) = F(g)F(f) \quad \text{and} \quad F(\text{id}_X) = \text{id}_{F(X)}$$