Universal Hinge Patterns for Folding Strips Efficiently into Any Grid Polyhedron

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Abstract. We present two universal hinge patterns that enable a strip of material to fold into any connected surface made up of unit squares on the 3D cube grid—for example, the surface of any polycube. The folding is efficient: for target surfaces topologically equivalent to a sphere, the strip needs to have only twice the target surface area, and the folding stacks at most two layers of material anywhere. These geometric results offer a new way to build programmable matter that is substantially more efficient than what is possible with a square $N \times N$ sheet of material, which can fold into all polycubes only of surface area $O(N)$ and may stack $\Theta(N^2)$ layers at one point. We also show how our strip foldings can be executed by a rigid motion without collisions (albeit assuming zero thickness), which is not possible in general with 2D sheet folding.

To achieve these results, we develop new approximation algorithms for milling the surface of a grid polyhedron, which simultaneously give a 2-approximation in tour length and an $8/3$-approximation in the number of turns. Both length and turns consume area when folding a strip, so we build on past approximation algorithms for these two objectives from 2D milling.

1 Introduction

In computational origami design, the goal is generally to develop an algorithm that, given a desired shape or property, produces a crease pattern that folds into an origami with that shape or property. Examples include folding any shape [9], folding approximately any shape while being watertight [10], and optimally folding a shape whose projection is a desired metric tree [14,15]. In all of these results, every different shape or tree results in a completely different crease pattern; two shapes rarely share many (or even any) creases.

The idea of a universal hinge pattern [6] is that a finite set of hinges (possible creases) suffice to make exponentially many different shapes. The main result along these lines is that an $N \times N$ “box-pleat” grid suffices to make any polycube

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made of $O(N)$ cubes [6]. The box-pleat grid is a square grid plus alternating diagonals in the squares, also known as the “tetrakis tiling”. For each target polycube, a subset of the hinges in the grid serve as the crease pattern for that shape. Polycubes form a universal set of shapes in that they can arbitrarily closely approximate (in the Hausdorff sense) any desired volume.

The motivation for universal hinge patterns is the implementation of programmable matter—material whose shape can be externally programmed. One approach to programmable matter, developed by an MIT–Harvard collaboration, is a self-folding sheet—a sheet of material that can fold itself into several different origami designs, without manipulation by a human origamist [12,1]. For practicality, the sheet must consist of a fixed pattern of hinges, each with an embedded actuator that can be programmed to fold or not. Thus for the programmable matter to be able to form a universal set of shapes, we need a universal hinge pattern.

The box-pleated polycube result [6], however, has some practical limitations that prevent direct application to programmable matter. Specifically, using a sheet of area $\Theta(N^2)$ to fold $N$ cubes means that all but a $\Theta(1/N)$ fraction of the surface area is wasted. Unfortunately, this reduction in surface area is necessary for a roughly square sheet, as folding a $1 \times 1 \times N$ tube requires a sheet of diameter $\Omega(N)$. Furthermore, a polycube made from $N$ cubes can have surface area as low as $\Theta(N^{2/3})$, resulting in further wastage of surface area in the worst case. Given the factor-$\Omega(N)$ reduction in surface area, an average of $\Omega(N)$ layers of material come together on the polycube surface. Indeed, the current approach can have up to $\Theta(N^2)$ layers coming together at a single point [6]. Real-world robotic materials have significant thickness, given the embedded actuation and electronics, meaning that only a few overlapping layers are really practical [12].

Our results: strip folding. In this paper, we introduce two new universal hinge patterns that avoid these inefficiencies, by using sheets of material that are long only in one dimension (“strips”). Specifically, Fig. 1 shows the two hinge patterns: the canonical strip is a $1 \times N$ strip with hinges at integer grid lines and same-oriented diagonals, while the zig-zag strip is an $N$-square zig-zag with hinges at just integer grid lines. We show in Section 2 that any grid surface—any connected surface made up of unit squares on the 3D cube grid—can be folded from either strip. The strip length only needs to be a constant factor larger than the surface area, and the number of layers is at most a constant throughout the folding. Most of our analysis concerns (genus-0) grid polyhedra, that is, when the surface is topologically equivalent to a sphere (a manifold without boundary, so that every edge is incident to exactly two grid squares, and without handles, unlike a torus). We show in Section 4 that a grid polyhedron of