Chapter One: Area and Dissection

Section 1. Qualitative and Quantitative Treatments of Area

We shall assume that the readers are familiar with simple terms such as triangles, squares, rectangles and parallelograms, simple concepts such as congruence, similarity, convexity and parallelism, and simple results such as the sum of the angles of a triangle being 180°. On the other hand, we will make precise the assumptions about area.

Our focus is on the area of polygons. The first assumption is called the existence of area, which states that every polygon has a non-negative area. This may seem unnecessarily legalistic, but it provides us with a context. With that in mind, we will safely ignore it from now on.

The next two assumptions allow us to give a qualitative treatment of area.

The Principle of Conservation of Area.
If a plane figure is dissected into several pieces, then its area is equal to the sum of the areas of the pieces.

The Principle of Preservation of Area.
If a plane figure is transferred to another location by rigid motion, then its area is unchanged.

In other words, congruent figures have equal area.

We start with a simple example. When a parallelogram is divided by either diagonal into two triangles, they have equal area because they are congruent.

Let $E$ be a point on the side $AB$ of a parallelogram $ABCD$. Is the area of triangle $CDE$ greater than, equal to or less than one half the area of $ABCD$?

![Figure 1.1](image-url)

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Draw a line through $E$ parallel to $AD$, cutting $CD$ at $F$, as shown in Figure 1.1. Then $EF$ divides $ABCD$ into two parallelograms $AEFD$ and $BEFC$. Now triangle $DEF$ has half the area of $AEFD$ while triangle $CEF$ has half the area of $BEFC$. By the Principle of Conservation of area, triangle $CDE$ has half the area of $ABCD$.

$P$ is a point inside a square $ABCD$. Which if either have greater total area, triangles $PAD$ and $PCB$, or triangles $PAB$ and $PCD$?

![Figure 1.2](image)

Draw a line through $P$ parallel to $AD$, cutting $AB$ at $E$ and $CD$ at $F$, as shown in Figure 1.2. Then $PAD$ has half the area of $ADFE$ and $PCB$ has half the area of $BCFE$. Together they have half the area of $ABCD$, and their total area is equal to the total area of $PAB$ and $PCD$.

$P$ and $R$ are points inside a rectangle $ABCD$ such that $RA$ cuts $PB$ at $Q$ and $RD$ cuts $PC$ at $S$, as shown in Figure 1.3. Prove that the total area of triangles $PAQ$ and $PDS$ is equal to the total area of triangles $RBQ$ and $RCS$.

![Figure 1.3](image)

When triangles $ABQ$ and $CDS$ are added to both side, the augmented total area of each side is now half the area of $ABCD$. Hence the total area of each side before augmentation must also be the same.