The Smooth Hom-Stack of an Orbifold

David Michael Roberts and Raymond F. Vozzo

Abstract For a compact manifold M and a differentiable stack $\mathcal{X}$ presented by a Lie groupoid X, we show the Hom-stack $\underline{\mathcal{H}\mathrm{om}}(M,\mathcal{X})$ is presented by a Fréchet–Lie groupoid $\mathrm{Map}(M,X)$ and so is an infinite-dimensional differentiable stack. We further show that if $\mathcal{X}$ is an orbifold, presented by a proper étale Lie groupoid, then $\mathrm{Map}(M,X)$ is proper étale and so presents an infinite-dimensional orbifold.

This note serves to announce a generalisation of the authors' work [8], which showed that the smooth loop stack of a differentiable stack is an infinite-dimensional differentiable stack, to more general mapping stacks where the source stack is a compact manifold (or more generally a compact manifold with corners). We apply this construction to differentiable stacks that are smooth orbifolds, that is, they can be presented by proper étale Lie groupoids (see Definition 9).

Existing work on mapping spaces of orbifolds has been considered in the case of C^k maps [4], of Sobolev maps [10] and smooth maps [3]; in the latter case several different notions of smooth orbifold maps are considered, from the point of view of orbifolds described by orbifold charts. In all these cases, some sort of orbifold structure has been found (for instance, Banach or Fréchet orbifolds).

Noohi [6] solved the problem of constructing a topological mapping stack between more general *topological* stacks, when the source stack has a presentation by a compact topological groupoid. See [8] for further references and discussion.

We take as given the definition of Lie groupoid in what follows, using finite-dimensional manifolds unless otherwise specified. Manifolds will be considered as trivial groupoids without comment. We pause only to note that in the infinite-dimensional setting, the source and target maps of Fréchet–Lie groupoids must be submersions between Fréchet manifolds, which is a stronger hypothesis than asking the derivative is surjective (or even split) everywhere, as in the finite-dimensional or Banach case.

D.M. Roberts (✉) • R.F. Vozzo
School of Mathematical Sciences, University of Adelaide, Adelaide, SA 5005, Australia
e-mail: david.roberts@adelaide.edu.au; raymond.vozzo@adelaide.edu.au

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We will also consider groupoids in diffeological spaces. Diffeological spaces (see e.g. [1]) contain Fréchet manifolds as a full subcategory and admit all pullbacks (in fact all finite limits) and form a cartesian closed category such that for K and M smooth manifolds with K compact, the diffeological mapping space M^K is isomorphic to the Fréchet manifold of smooth maps $K \to M$.

Differentiable stacks are, for us, stacks of groupoids on the site $\mathcal{M}$ of finite-dimensional smooth manifolds with the open cover topology that admit a presentation by a Lie groupoid [2]. We can also consider the more general notion of stacks that admit a presentation by a diffeological or Fréchet–Lie groupoid.

Definition 1 Let $\mathcal{X}, \mathcal{Y}$ be stacks on $\mathcal{M}$. The *Hom-stack* $\underline{\mathcal{H}\mathrm{om}}(\mathcal{Y}, \mathcal{X})$ is defined by taking the value on the manifold N to be $\mathbf{Stack}_{\mathcal{M}}(\mathcal{Y} \times N, \mathcal{X})$.

Thus we have a Hom-stack for any pair of stacks on $\mathcal{M}$. The case we are interested in is where we have a differentiable stack $\mathcal{X}$ associated to a Lie groupoid X, e.g. an orbifold, and the resulting Hom-stack $\underline{\mathcal{H}\mathrm{om}}(M, \mathcal{X})$ for M a compact manifold.

We define a *minimal cover* of a manifold M to be a cover by regular closed sets V_i such that the interiors $V_i{}^{\mathrm{o}}$ form an open cover of M, and every $V_i{}^{\mathrm{o}}$ contains a point not in any other $V_j{}^{\mathrm{o}}$. We also ask that finite intersections $V_i \cap \ldots \cap V_k$ are also regular closed. Denote the collection of minimal covers of a manifold M by $C(M)_{\min}$, and note that such covers are cofinal in open covers. Recall that a cover V of a manifold defines a diffeological groupoid $\check{C}(V)$ with objects $\coprod_i V_i$ and arrows $\coprod_{i,j} V_i \cap V_j$.[1] We are particularly interested in the case when we take the closure $\{\overline{U_i}\}$ of $\{U_i\}$, a good open cover, minimal in the above sense.

We denote the arrow groupoid of a Lie groupoid X by $X^{\mathbf{2}}$—it is again a Lie groupoid and comes with functors $S, T \colon X^{\mathbf{2}} \to X$, with object components source and target, resp. Let M be a compact manifold with corners and X a Lie groupoid. Define the *mapping groupoid* $\mathrm{Map}(M, X)$ to be the following diffeological groupoid. The object space $\mathrm{Map}(M, X)_0$ is the disjoint union over minimal covers V of the spaces $X^{\check{C}(V)}$ of functors $\check{C}(V) \to X$. The arrow space $\mathrm{Map}(M, X)_1$ is

$$\coprod_{V_1, V_2 \in C(M)_{\min}} X^{\check{C}(V_1)} \times_{X^{\check{C}(V_{12})}} (X^{\mathbf{2}})^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)}$$

where the chosen minimal refinement $V_{12} \subset V_1 \times_M V_2$ is defined using the boolean product on the algebra of regular closed sets. The maps

$$S, T \colon (X^{\mathbf{2}})^{\check{C}(V_{12})} \to X^{\check{C}(V_{12})} \qquad \text{and} \qquad X^{\check{C}(V_i)} \to X^{\check{C}(V_{12})} \quad (i = 1, 2) \tag{1}$$

give us a pullback and the two projections

$$X^{\check{C}(V_1)} \times_{X^{\check{C}(V_{12})}} (X^{\mathbf{2}})^{\check{C}(V_{12})} \times_{X^{\check{C}(V_{12})}} X^{\check{C}(V_2)} \longrightarrow X^{\check{C}(V_i)}, \tag{2}$$

[1] We can in what follows safely ignore the issue of intersections of boundaries.