Chapter 6

Variations of the Puzzle

TH is an example of a one person game; such games are known as solitaire games. There are plenty of other mathematical solitaire games, the Icosian game, the Fifteen puzzle, and Rubik’s Cube are just a few prominent examples. Numerous variations of the TH can also be defined, some natural and some not that natural. In fact, Lucas himself in [286, p. 303] pointed out the following: “Le nombre des problèmes que l’on peut se poser sur la nouvel Tour d’Hanoï est incalculable.”  

Many variations were indeed studied and some of them we already encountered in previous chapters: the Linear TH in Chapter 2, problems in Chapter 3 allowing for irregular states, the Switching TH in Chapter 4, and the tasks in Chapter 5 where more than three pegs are available.

In the next section we make clear what is understood as a variant of the TH. We first illustrate this by a brief look at a variety called Exchanging Discs TH, and then by introducing and solving the Black and White TH. Numerous “colored” variants are listed in the subsequent section, including the Tower of Antwerpen. In the concluding section we present the Bottleneck TH which allows for larger discs above smaller ones up to a certain discrepancy. We describe an optimal algorithm and note that the solution for the Bottleneck TH may not be unique. We close this chapter by briefly mentioning a related version, the Sinner’s TH.

6.1 What is a Tower of Hanoi Variant?

In order to make precise which variants of the TH are of interest, we define the framework as follows.

Any variant of the TH consists of pegs and discs such that discs can be stacked onto pegs. In addition, it obeys the following common rules:

1. Pegs are distinguishable.

The number of problems which one can pose oneself on the new Tower of Hanoï is incalculable.
2. Discs are distinguishable.

3. Discs are on pegs all the time except during moves.

4. One or more discs can only be moved from the top of a stack.

5. Task: given an initial distribution of discs among pegs (initial state) and a goal distribution of discs among pegs (final state), find a shortest sequence of moves that transfers discs from the initial state to the final state obeying the rules.

Although the above conventions appear rather restrictive, they offer a tremendous number of different variations. For instance,

- there can be an arbitrary number of pegs (as we have seen already);
- pegs can be distinguished also by their heights, that is, by the number of discs they can hold;
- discs can be distinguished in size and/or color;
- (certain) irregular (with respect to TH rules) states may be admitted;
- more than one top disc may be moved in a single move;
- there can be additional restrictions or relaxations on moves, the latter even violating the divine rule;
- and, of course, any combination of the above.

The interest for such variants goes all the way back to Lucas, who in [286] proposed a variant with five pegs and four groups of discs of different colors. Only the central peg can hold all discs; see Figure 0.11. Every group contains four discs and the 16 discs have pairwise different sizes. The group of color \( c \in \{4\} \) consists of the four discs \( d \in \{16\} \) with \( 1 + (16 - d) \mod 4 = c \). All this can be deduced from the figure of the goal state as produced by Lucas on the first page of his article. Figure 6.1 shows the initial configuration of this type of puzzle. A possible task is to transfer all discs onto the middle peg obeying the divine rule; see [285, Quatrième problème].

Note that since the 16 discs are of mutually different sizes, this puzzle is equivalent to an instance of a type P1 problem (that is, to reach a perfect state from a given regular state) with \( p = 5 \) and \( n = 16 \), namely, \( (1234)^4 \to 0^{16} \). In [285], Lucas also proposed type P2 problems (to reach a regular from another regular state). Although his formulation of the tasks leaves room for interpretation, we think that he is requiring two [285, Première problème] or three [285, Troisième problème] colors to be united on the center peg, respectively, with the other colors remaining in their initial position. Finally, Lucas proposes similar tasks with the initial state changed to \( 1^42^43^44^4 \), i.e. the discs grouped in four towers according to size, and the inverse problems.