1 Ordinary Differential Equations

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1.1 INTRODUCTION

Differential equations* play a vital role in the solutions to many problems encountered when modeling physical phenomena. All the disciplines in the physical sciences, with their own unique interests representing a variety of physical situations, require that the student be able to derive the necessary mathematical equation (often a differential equation) and then solve the equation to obtain the desired solution. We shall consider a variety of physical situations that lead to differential equations, using representative problems from several disciplines, and standard methods used to solve the equations will be developed.

*It is assumed that a course in differential equations precedes the course in which this text is used. This chapter is meant to be a quick review of the parts of significance to the science student.
An equation involving one or more derivatives of a function is a differential equation. The solution of a differential equation is an expression involving the dependent and independent variables, free of derivatives and integrals, which when substituted back into the differential equation results in an identity. The solution is valid in some domain in which it is defined and differentiable; however, it may or may not be stable. If it is unstable, it may lead to a second solution. Questions of stability will not be considered in this book. Often, exact solutions are difficult, if not impossible, to determine, and then approximate solutions are sought or the problem is solved on the computer through the use of numerical methods such as MATLAB. We shall present some exact solutions in this chapter and several numerical methods in Chapter 8. A textbook on differential equations should be sought for completeness.

1.2 DEFINITIONS

An ordinary differential equation is one in which only total derivatives appear. A partial differential equation is one that involves partial derivatives. If a dependent variable is a function of only one independent variable, such as \( f(x) \), an ordinary differential equation would result; however, if the dependent variable depends on more than one independent variable, such as \( f(x, y, z) \), a partial differential equation may describe the phenomenon of interest (see Chapter 6).

The dependent variable is usually the unknown quantity sought after in a problem, or it leads directly to the desired quantity. For example, the lift on an airfoil is the quantity desired; to determine the lift we would solve a partial differential equation to find the unknown velocity \( v(x, y) \), from which we could calculate the pressure and subsequently the desired lift.

The order of a differential equation is equal to the order of the highest derivative. An equation is linear if it contains only terms of the first degree in the dependent variable and its derivatives; if it contains a term that involves combinations of derivatives or products of the dependent variable, it is nonlinear. A differential equation is homogeneous if it can be written in a form such that all terms contain the dependent variable or one of its derivatives. The equation

\[
x^2 \frac{d^2 f}{dx^2} + x \frac{df}{dx} + (x^2 - 4)f = 0
\]

(1.2.1)

is a linear, homogeneous, ordinary differential equation of second order. The equation

\[
\frac{\partial^4 u}{\partial x^4} + 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0
\]

(1.2.2)

is a linear, homogeneous, partial differential equation of fourth order. The equation

\[
\frac{d^2 f}{dx^2} + 4f \frac{df}{dx} + 2f = \cos x
\]

(1.2.3)

is a nonlinear, nonhomogeneous, ordinary differential equation of second order. The degree of an equation is the degree of the highest ordered derivative that occurs, if the derivatives can be written in polynomial form; for example, \( \sin(d^2 f/dx^2) \) has no degree, whereas \( (df/dx)^2 \) is of degree 2.