ADMISSIBILITY CONDITIONS FOR WEAK SOLUTIONS OF NONSTRICLY HYPERBOLIC SYSTEMS.

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SUMMARY

We discuss questions related to the choice of a proper class of waves and initial conditions for the well-posedness of the Riemann problem for nonstrictly hyperbolic systems of conservation laws. Since multiple eigenvalues represent strong or resonant wave interaction we propose to derive a relatively simple and universal set of model equations which describe qualitatively the underlying processes, like Burgers’ equation does in a strictly hyperbolic case. Finally, we discuss numerical schemes utilizing various Riemann solvers to the above class of problems.

INTRODUCTION

Consider the following hyperbolic system of conservation laws

\[ U_t + (F(U))_x = 0, \]

where \( U \) represents vector of dependent variables and \( F(U) \) denotes the appropriate flux function. Riemann problem is an initial value problem with a piecewise constant initial data. It serves as an intermediate step in understanding of the general initial value problem and as an important model problem in various applications, such as piston and shock-tube problems in gas dynamics, Riemann solvers in numerical schemes, simplest model of the field line reconnection, etc.

Solution is understood in a weak sense and consists of combinations of various waves propagating out from the initial discontinuity. There are two major problems:

a) to prove well-posedness (for example, existence, uniqueness, and stability) of the problem in some class of waves,

b) to choose the class of waves by studying physical equations on the next smaller length scale (for example, to introduce back into equations diffusive, dispersive, forcing or any other terms, which were dropped in the first place).

If the eigenvalues of the Jacobian matrix may coincide, then the appropriate system of conservation laws is called nonstrictly hyperbolic. Application of
standard admissibility conditions worked out for a strictly hyperbolic case often leads to contradictions and various difficulties arising from the fact that those systems are fundamentally different. Physically, coinciding eigenvalues represent strong or resonant wave interaction compared to weak interactions represented by strictly hyperbolic systems. Mathematically, this manifests itself in the existence of free parameters for the solutions with discontinuous initial data, because interaction parameters are "hidden" in the discontinuity and are not introduced explicitly into the problem. Numerically, the discontinuous initial data produces different results depending on the numerical schemes used since the interaction parameters are introduced by truncation errors of the appropriate schemes (see article of B. Wendroff in this volume for additional illustration of this problem arising in applications of irregular grids).

For many systems nonstrict hyperbolicity implies nonconvexity, for example, the steepening rate of smooth waves may be zero at some points. For an example of such systems see a paper of H. Freistüler in this volume.

EXAMPLES OF THE EQUATIONS

One-dimensional equations of ideal magnetohydrodynamics (MHD) characterize the flow of conducting fluid in the presence of magnetic field and represent coupling of the fluid dynamical equations with Maxwell's equations of electrodynamics. Neglecting displacement current, electrostatic forces, effects of viscosity, resistivity, and heat conduction, one obtains the following ideal MHD equations [1]:

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + P^*)_x &= 0, \\
(\rho v)_t + (\rho uv - B_0 B)_x &= 0, \\
(\rho w)_t + (\rho uw - B_0 H)_x &= 0, \\
B_t + (Bu - B_0 v)_x &= 0, \\
H_t + (Hu - B_0 w)_x &= 0, \\
E_t + ((E + P^*)u - B_0 (B_0 u + Bv + Hw))_x &= 0.
\end{align*}
\]

In the above equations, the following notations are used: \(\rho\) for density, \(\mathbf{u} = (u, v, w)\) for velocity, \(\mathbf{B} = (B_0, B, H)\) for magnetic field, \(P\) for static pressure, \(P^*\) for full pressure, \(P^* = P + \frac{1}{2}|\vec{B}|^2\), \(E\) for energy, \(E = \frac{1}{2}\rho |\vec{u}|^2 + P/\gamma - 1 + |\vec{B}|^2\), \(\gamma\) for ratio of specific heats, and \(B_0 \equiv \text{const}\).

The eigenvalues of the Jacobian matrix can be written in nondecreasing order as

\(u - c_f, \ u - c_a, \ u - c_s, \ u, \ u + c_a, \ u + c_s, \ u + c_f,\)