COMPUTATION OF STEADY EULER EQUATIONS USING FINITE ELEMENT METHOD

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SUMMARY

A Clebsch variable form of the steady Euler equations is solved using the finite element method. The method is applied to the solution of two-dimensional transonic Euler equations for presentation at the 1986 GAMM Workshop. The numerical results include transonic flow around a circular cylinder at \( M_\infty = 0.50 \), lifting transonic flow around NACA0012 airfoil at \( M_\infty = 0.85 \) and \( \beta_\infty = 1.0^\circ \), and a supercritical flow around Korn airfoil at \( M_\infty = 0.75 \).

METHOD OF ANALYSIS

The details of the method of analysis used here, in solving transonic Euler equations, have been documented elsewhere, e.g., references 1 - 5. Only a summary of the method will be presented here.

When a Clebsch transformation of the velocity vector, \( \mathbf{u} \), in the form

\[
\mathbf{u} = \nabla \phi + \alpha \nabla \beta ,
\]

is used, the Euler equations governing steady, inviscid and isoenergetic flows can be expressed as follows:

\[
\nabla \cdot \left[ \rho (\nabla \phi + \alpha \nabla \beta) \right] = 0 ,
\]

\[
\rho u \cdot \nabla \beta = - \frac{p}{R} S,\alpha ,
\]

\[
\rho u \cdot \nabla \alpha = 0 .
\]

In Equations (1) - (4), \( \phi \) is a velocity potential function, \( \alpha \) is a material coordinate, \( \beta \) is a Lagrangian multiplier of a variational principle associated with the corresponding flow equations. Also, \( \rho \) is mass density, \( p \) is pressure, \( S \) is entropy and \( R \) is constant.

Using the Clebsch variables \( \phi,\alpha,\beta \) and the equation of state

\[
p = \kappa \rho \gamma (\gamma - 1) S/R ,
\]

where for inviscid flows \( S = S(\alpha) \), it can be verified that Equations (2) - (4) are equivalent to a more familiar form of the Euler equations expressed...
in terms of primitive variables: \( u, \rho \) and \( S \). More specifically, Equation (2) is the conservation of mass, Equation (3) is equivalent to the momentum equations and Equation (4) is known as the conservation equation for identity of particles. The vorticity vector \( \xi \) with the Clebsch variables takes the form:

\[
\xi = v \times \xi
\]  

(6)

The above set of Euler equations requires the specification of the normal mass flux on all boundaries, and the specification of \( \alpha \) and \( \beta \) at the freestream inflow boundaries. For uniform inflow conditions, \( \beta \) and \( S \) are set equal to zero, and \( \alpha \) is set to \( y \). The entropy distribution \( S = S(\alpha) \) after a shock wave is determined from the Rankine-Hugoniot shock jump conditions, expressed in the form:

\[
[S] = \frac{R}{(\gamma - 1)} \left[ \log_e \left( 1 + \frac{2\gamma}{\gamma + 1}(M_n^2 - 1) \right) - \gamma \log_e \frac{(\gamma + 1)M_n^2}{(\gamma - 1)M_n^2 + 2} \right] > 0 ,
\]  

(7)

where \([S] = S^+ - S^-\), and \( M_n \) is the normal component of the upstream Mach number \( M^+ \) at the shock.

For obtaining numerical solutions, Equations (3) and (4) are cast into forms with second-order derivatives of \( \beta \) and \( \alpha \), and take the following form:

\[
\rho u \cdot \nabla (\rho u \cdot \nabla \beta) = -\rho u \cdot \nabla (\frac{\partial}{\partial \xi} S, \alpha) , \quad (8)
\]

\[
\rho u \cdot \nabla (\rho u \cdot \nabla \alpha) = 0 . \quad (9)
\]

This way, all Clebsch variables have second-order derivatives.

From the equation of state, the mass density and the static pressure can further be expressed, respectively, as in the following:

\[
\rho = c_0 (H - \frac{1}{2}u \cdot u)^{\Theta - 1} e^{-S/R} , \quad (10)
\]

\[
p = c (H - \frac{1}{2}u \cdot u)^{\Theta} e^{-S/R} , \quad (11)
\]

where \( H \) is the stagnation enthalpy (a constant) and

\[
\Theta = \gamma/(\gamma - 1), \quad c = \kappa (\kappa_0)^{-\Theta} . \quad (12)
\]

In the isentropic flow regions, the mass density and the static pressure do not vary with the entropy, thus \( S \) is set to zero in Equations (10) and (11).