Conjectures about Discriminants of Hecke Algebras of Prime Level

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Abstract. In this paper, we study \( p \)-divisibility of discriminants of Hecke algebras associated to spaces of cusp forms of prime level. By considering cusp forms of weight bigger than 2, we are led to make a precise conjecture about indexes of Hecke algebras in their normalisation which implies (if true) the surprising conjecture that there are no mod \( p \) congruences between non-conjugate newforms in \( S_2(\Gamma_0(p)) \), but there are almost always many such congruences when the weight is bigger than 2.

1 Basic Definitions

We first recall some commutative algebra related to discriminants, then introduce Hecke algebras of spaces of cusp forms.

1.1 Commutative Algebra

In this section we recall the definition of discriminant of a finite algebra and note that the discriminant is nonzero if and only if no base extension of the algebra contains nilpotents.

Let \( R \) be a ring and let \( A \) be an \( R \)-algebra that is free of finite rank as an \( R \)-module. The trace of \( x \in A \) is the trace, in the sense of linear algebra, of left multiplication by \( x \).

Definition 1 (Discriminant). Let \( \omega_1, \ldots, \omega_n \) be an \( R \)-basis for \( A \). Then the discriminant \( \text{disc}(A) \) of \( A \) is the determinant of the \( n \times n \) matrix \( (\text{tr}(\omega_i \omega_j)) \).

The discriminant is only well-defined modulo squares of units in \( R \). When \( R = \mathbb{Z} \) the discriminant is well defined, since the only units are \( \pm 1 \).

We say that \( A \) is separable over \( R \) if for every extension \( R' \) of \( R \), the ring \( A \otimes R' \) contains no nilpotents.

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**Proposition 1.** Suppose $R$ is a field. Then $A$ has nonzero discriminant if and only if $A$ is separable over $R$.

**Proof.** For the convenience of the reader, we summarize the proof in [Mat86, §26]. If $A$ contains a nilpotent then that nilpotent is in the kernel of the trace pairing, so the discriminant is 0. Conversely, if $A$ is separable then we may assume that $R$ is algebraically closed. Then $A$ is an Artinian reduced ring, hence isomorphic as a ring to a finite product of copies of $R$, since $R$ is algebraically closed. Thus the trace form on $A$ is nondegenerate.

### 1.2 The Discriminant Valuation

We next introduce Hecke algebras attached to certain spaces of cusp forms of prime level $p$, define the discriminant valuation as the exponent of the largest power of $p$ that divides the discriminant, and observe that there are eigenform congruences modulo $p$ exactly when the discriminant valuation is positive. We then present an example to illustrate the definitions.

Let $\Gamma$ be a congruence subgroup of $\text{SL}_2(\mathbb{Z})$. In this paper, we will only consider $\Gamma = \Gamma_0(p)$ for $p$ prime. For any positive integer $k$, let $S_k(\Gamma)$ denote the space of holomorphic weight $k$ cusp forms for $\Gamma$. Let

$$T = \mathbb{Z}[\ldots, T_n, \ldots] \subset \text{End}(S_k(\Gamma))$$

be the associated Hecke algebra, which is generated by Hecke operators $T_n$ for all integers $n$, including $n = p$ (we will sometimes write $U_p$ for $T_p$). Then $T$ is a commutative ring that is free as a module over $\mathbb{Z}$ of rank equal to $\dim S_k(\Gamma)$. We will also sometimes consider the image $T^{\text{new}}$ of $T$ in $\text{End}(S_k(\Gamma)^{\text{new}})$.

**Definition 2 (Discriminant Valuation).** Let $p$ be a prime, $k$ a positive integer, and suppose that $\Gamma = \Gamma_0(p)$. Let $T$ be the corresponding Hecke algebra. Then the discriminant valuation of $\Gamma$ in weight $k$ is

$$d_k(\Gamma) = \text{ord}_p(\text{disc}(T)).$$

We expect that $d_k(\Gamma)$ is finite for the following reason. The Hecke operators $T_n$, with $n$ not divisible by $p$, are diagonalizable since they are self adjoint with respect to the Petersson inner product. When $k = 2$ one knows that $U_p$ is diagonalizable since the level is square free, and when $k > 2$ one expects this (see [CE98]). If $T$ contains no nilpotents, Proposition 1 implies that the discriminant of $T$ is nonzero. Thus $d_k(\Gamma)$ is finite when $k = 2$ and conjectured to be finite when $k > 2$.

Let $p$ be a prime and suppose that $\Gamma = \Gamma_0(p)$. A *normalised eigenform* is an element $f = \sum a_n q^n \in S_k(\Gamma)$ that is an eigenvector for all Hecke operators $T_\ell$, including those that divide $p$, normalised so that $a_1 = 1$. The quantity $d_k(\Gamma)$ is of interest because it measures mod $p$ congruences between normalised eigenforms in $S_k(\Gamma)$. 

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