Chapter 10

Difference equations

10.1	$x_t = a_t x_{t-1} + b_t, \qquad t = 1, 2, \ldots$	A *first-order linear difference equation.*
10.2	$x_t = \left(\prod_{s=1}^{t} a_s\right) x_0 + \sum_{k=1}^{t} \left(\prod_{s=k+1}^{t} a_s \right) b_k$	The solution of (10.1) if we define the "empty" product $\prod_{s=t+1}^{t} a_s$ as 1.
10.3	$x_t = a^t x_0 + \sum_{k=1}^{t} a^{t-k} b_k, \qquad t = 1, 2, \ldots$	The solution of (10.1) when $a_t = a$, a constant.
10.4	• $x_t = Aa^t + \sum_{s=0}^{\infty} a^s b_{t-s}, \quad \|a\| < 1$ • $x_t = Aa^t - \sum_{s=1}^{\infty} \left(\frac{1}{a}\right)^s b_{t+s}, \quad \|a\| > 1$	The *backward* and *forward solutions* of (10.1), respectively, with $a_t = a$, and with A as an arbitrary constant.
10.5	$x_t = a x_{t-1} + b \Leftrightarrow x_t = a^t \left(x_0 - \frac{b}{1-a} \right) + \frac{b}{1-a}$	Equation (10.1) and its solution when $a_t = a \neq 1$, $b_t = b$.
10.6	$(*) \quad x_t + a_1(t) x_{t-1} + \cdots + a_n(t) x_{t-n} = b_t$ $(**) \quad x_t + a_1(t) x_{t-1} + \cdots + a_n(t) x_{t-n} = 0$	$(*)$ is the *general linear inhomogeneous difference equation of order* n, and $(**)$ is the associated *homogeneous* equation.
10.7	If $u_1(t), \ldots, u_n(t)$ are linearly independent solutions of (10.6) $(**)$, u_t^* is some particular solution of (10.6) $(*)$, and $C_1, \ldots, C_n$ are arbitrary constants, then the general solution of $(**)$ is $x_t = C_1 u_1(t) + \cdots + C_n u_n(t)$ and the general solution of $(*)$ is $x_t = C_1 u_1(t) + \cdots + C_n u_n(t) + u_t^*$	The structure of the solutions of (10.6). (For linear independence, see (11.21).)

10.8 For $b \neq 0$, $x_t + ax_{t-1} + bx_{t-2} = 0$ has the solution:

- For $\frac{1}{4}a^2 - b > 0:\ x_t = C_1 m_1^t + C_2 m_2^t$, where $m_{1,2} = -\frac{1}{2}a \pm \sqrt{\frac{1}{4}a^2 - b}$.
- For $\frac{1}{4}a^2 - b = 0:\ x_t = (C_1 + C_2 t)(-a/2)^t$.
- For $\frac{1}{4}a^2 - b < 0:\ x_t = Ar^t \cos(\theta t + \omega)$, where $r = \sqrt{b}$ and $\cos\theta = -\dfrac{a}{2\sqrt{b}}$, $\theta \in [0, \pi]$.

The solutions of a homogeneous, linear second-order difference equation with constant coefficients a and b. C_1, C_2, and ω are arbitrary constants.

10.9 To find a particular solution of

$$(*) \quad x_t + ax_{t-1} + bx_{t-2} = c_t, \quad b \neq 0$$

use the following trial functions and determine the constants by using the method of undetermined coefficients:

- If $c_t = c$, try $u_t^* = A$.
- If $c_t = ct + d$, try $u_t^* = At + B$.
- If $c_t = t^n$, try $u_t^* = A_0 + A_1 t + \cdots + A_n t^n$.
- If $c_t = c^t$, try $u_t^* = Ac^t$.
- If $c_t = \alpha \sin ct + \beta \cos ct$, try $u_t^* = A \sin ct + B \cos ct$.

If the function c_t is itself a solution of the homogeneous equation, multiply the trial solution by t. If this new trial function also satisfies the homogeneous equation, multiply the trial function by t again. (See Hildebrand (1968), Sec. 1.8 for the general procedure.)

10.10

$$\begin{aligned} (*) &\quad x_t + a_1 x_{t-1} + \cdots + a_n x_{t-n} = b_t \\ (**) &\quad x_t + a_1 x_{t-1} + \cdots + a_n x_{t-n} = 0 \end{aligned}$$

Linear difference equations with constant coefficients.

10.11

$$m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0$$

The *characteristic equation* of (10.10). Its roots are called *characteristic roots*.

10.12 Suppose the characteristic equation (10.11) has n different roots, $\lambda_1, \ldots, \lambda_n$, and define

$$\theta_r = \frac{\lambda_r}{\prod\limits_{\substack{1 \le s \le n \\ s \neq r}} (\lambda_r - \lambda_s)}, \quad r = 1, 2, \ldots, n$$

Then a special solution of (10.10) $(*)$ is given by

$$u_t^* = \sum_{r=1}^{n} \theta_r \sum_{i=0}^{\infty} \lambda_r^i b_{t-i}$$

The *backward* solution of (10.10)$(*)$, valid if $|\lambda_r| < 1$ for $r = 1, \ldots, n$.