Chapter 11

Differential equations

First-order equations

11.1 $\dot{x}(t) = f(t) \iff x(t) = x(t_0) + \int_{t_0}^{t} f(\tau)\, d\tau$

A simple differential equation and its solution. $f(t)$ is a given function and $x(t)$ is the unknown function.

11.2 $$\frac{dx}{dt} = f(t)g(x) \iff \int \frac{dx}{g(x)} = \int f(t)\, dt$$

Evaluate the integrals. Solve the resulting implicit equation for $x = x(t)$.

A *separable* differential equation. If $g(a) = 0$, $x(t) \equiv a$ is a solution.

11.3 $\dot{x} = g(x/t)$ and $z = x/t \implies t\dfrac{dz}{dt} = g(z) - z$

A *projective* differential equation. The substitution $z = x/t$ leads to a separable equation for z.

11.4 The equation $\dot{x} = B(x-a)(x-b)$ has the solutions

$$x \equiv a, \quad x \equiv b, \quad x = a + \frac{b-a}{1 - Ce^{B(b-a)t}}$$

$a \neq b$. $a = 0$ gives the *logistic* equation. C is a constant.

11.5
- $\dot{x} + ax = b \Leftrightarrow x = Ce^{-at} + \dfrac{b}{a}$
- $\dot{x} + ax = b(t) \Leftrightarrow x = e^{-at}(C + \int b(t)e^{at}\, dt)$

Linear first-order differential equations with constant coefficient $a \neq 0$. C is a constant.

11.6 $\dot{x} + a(t)\,x = b(t) \iff$

$$x = e^{-\int a(t)\, dt}\left(C + \int e^{\int a(t)\, dt} b(t)\, dt\right)$$

General linear first-order differential equation. $a(t)$ and $b(t)$ are given. C is a constant.

11.7

$$\dot{x} + a(t)\,x = b(t) \iff$$
$$x(t) = x_0 e^{-\int_{t_0}^{t} a(\xi)\,d\xi} + \int_{t_0}^{t} b(\tau) e^{-\int_{\tau}^{t} a(\xi)\,d\xi}\,d\tau$$

Solution of (11.6) with given initial condition $x(t_0) = x_0$.

11.8

$\dot{x} + a(t)x = b(t)x^r$ has the solution

$$x(t) = e^{-A(t)}\Big[C + (1-r)\int b(t)e^{(1-r)A(t)}\,dt\Big]^{\frac{1}{1-r}}$$

where $A(t) = \int a(t)\,dt$.

Bernoulli's equation and its solution ($r \neq 1$). C is a constant. (If $r = 1$, the equation is separable.)

11.9

$$\dot{x} = P(t) + Q(t)\,x + R(t)\,x^2$$

Riccati's equation. Not analytically solvable in general. The substitution $x = u + 1/z$ works *if* we know a particular solution $u = u(t)$.

11.10

The differential equation

$$(*) \quad f(t,x) + g(t,x)\,\dot{x} = 0$$

is called *exact* if $f'_x(t,x) = g'_t(t,x)$. The solution $x = x(t)$ is then given implicitly by the equation $\int_{t_0}^{t} f(\tau,x)\,d\tau + \int_{x_0}^{x} g(t_0,\xi)\,d\xi = C$ for some constant C.

An *exact* equation and its solution.

11.11

A function $\beta(t,x)$ is an *integrating factor* for $(*)$ in (11.10) if $\beta(t,x)f(t,x) + \beta(t,x)g(t,x)\dot{x} = 0$ is exact.

- If $(f'_x - g'_t)/g$ is a function of t alone, then $\beta(t) = \exp[\int (f'_x - g'_t)/g\,dt]$ is an integrating factor.
- If $(g'_t - f'_x)/f$ is a function of x alone, then $\beta(x) = \exp[\int (g'_t - f'_x)/f\,dx]$ is an integrating factor.

Results which occasionally can be used to solve equation $(*)$ in(11.10).

11.12

Consider the *initial value problem*

$$(*) \quad \dot{x} = F(t,x), \quad x(t_0) = x_0$$

where $F(t,x)$ and $F'_x(t,x)$ are continuous over the rectangle

$$\Gamma = \big\{\,(t,x) : |t - t_0| \le a,\ |x - x_0| \le b\,\big\}$$

Define

$$M = \max_{(t,x)\in\Gamma} |F(t,x)|, \quad r = \min\big(a, b/M\big)$$

Then $(*)$ has a unique solution $x(t)$ on the open interval $(t_0 - r, t_0 + r)$, and $|x(t) - x_0| \le b$ in this interval.

A *(local) existence and uniqueness theorem.*