Chapter 12

Topology in Euclidean space

12.1 $B(\mathbf{a}; r) = \{\, \mathbf{x} : \|\mathbf{x} - \mathbf{a}\| < r \,\} \qquad (r > 0)$

Definition of an *open* n*-ball* with radius r and center $\mathbf{a}$ in $\mathbb{R}^n$. ($\|\ \|$ is defined in (18.13).)

12.2
- A point $\mathbf{a}$ in $S \subset \mathbb{R}^n$ is an *interior point* of S if there exists an n-ball with center at $\mathbf{a}$, all of whose points belong to S.
- A point $\mathbf{b} \in \mathbb{R}^n$ (not necessarily in S) is a *boundary point* of S if every n-ball with center at $\mathbf{b}$ contains at least one point in S and at least one point not in S.

Definition of interior points and boundary points.

12.3 A set S in $\mathbb{R}^n$ is called
- *open* if all its points are interior points,
- *closed* if $\mathbb{R}^n \setminus S$ is open,
- *bounded* if there exists a number M such that $\|\mathbf{x}\| \leq M$ for all $\mathbf{x}$ in S,
- *compact* if it is closed and bounded.

Important definitions. $\mathbb{R}^n \setminus S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \notin S\}$.

12.4 A set S in $\mathbb{R}^n$ is closed if and only if it contains all its boundary points. The set $\bar{S}$ consisting of S and all its boundary points is called the *closure* of S.

A useful characterization of closed sets, and a definition of the closure of a set.

12.5 A set S in $\mathbb{R}^n$ is called a *neighborhood* of a point $\mathbf{a}$ in $\mathbb{R}^n$ if $\mathbf{a}$ is an interior point of S.

Definition of a neighborhood.

12.6 A sequence $\{\mathbf{x}_k\}$ in $\mathbb{R}^n$ *converges* to $\mathbf{x}$ if for every $\varepsilon > 0$ there exists an integer N such that $\|\mathbf{x}_k - \mathbf{x}\| < \varepsilon$ for all $k \geq N$.

Convergence of a sequence in $\mathbb{R}^n$. If the sequence does not converge, it *diverges*.

12.7 A sequence $\{\mathbf{x}_k\}$ in $\mathbb{R}^n$ is a *Cauchy sequence* if for every $\varepsilon > 0$ there exists an integer N such that $\|\mathbf{x}_j - \mathbf{x}_k\| < \varepsilon$ for all $j, k \geq N$.

Definition of a Cauchy sequence.

12.8 A sequence $\{\mathbf{x}_k\}$ in $\mathbb{R}^n$ converges if and only if it is a Cauchy sequence.

Cauchy's convergence criterion.

12.9 A set S in $\mathbb{R}^n$ is closed if and only if the limit $\mathbf{x} = \lim_k \mathbf{x}_k$ of each convergent sequence $\{\mathbf{x}_k\}$ of points in S also lies in S.

Characterization of a closed set.

12.10 Let $\{\mathbf{x}_k\}$ be a sequence in $\mathbb{R}^n$, and let $k_1 < k_2 < k_3 < \cdots$ be an increasing sequence of integers. Then $\{\mathbf{x}_{k_j}\}_{j=1}^{\infty}$, is called a *subsequence* of $\{\mathbf{x}_k\}$.

Definition of a subsequence.

12.11 A set S in $\mathbb{R}^n$ is compact if and only if every sequence of points in S has a subsequence that converges to a point in S.

Characterization of a compact set.

12.12 A collection $\mathcal{U}$ of open sets is said to be an *open covering* of the set S if every point of S lies in at least one of the sets from $\mathcal{U}$. The set S has the *finite covering property* if whenever $\mathcal{U}$ is an open covering of S, then a finite subcollection of the sets in $\mathcal{U}$ covers S.

A useful concept.

12.13 A set S in $\mathbb{R}^n$ is compact if and only if it has the finite covering property.

The Heine–Borel theorem.

12.14 $f : M \subset \mathbb{R}^n \to \mathbb{R}$ is *continuous* at $\mathbf{a}$ in M if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(\mathbf{x}) - f(\mathbf{a})| < \varepsilon$$

for all $\mathbf{x}$ in M with $\|\mathbf{x} - \mathbf{a}\| < \delta$.

Definition of a continuous function of n variables.

12.15 The function $\mathbf{f} = (f_1, \dots, f_m) : M \subset \mathbb{R}^n \to \mathbb{R}^m$ is *continuous* at a point $\mathbf{a}$ in M if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})\| < \varepsilon$$

for all $\mathbf{x}$ in M with $\|\mathbf{x} - \mathbf{a}\| < \delta$.

Definition of a continuous vector function of n variables.

12.16 Let $\mathbf{f} = (f_1, \dots, f_m)$ be a function from $M \subset \mathbb{R}^n$ into $\mathbb{R}^m$, and let $\mathbf{a}$ be a point in M. Then:

- $\mathbf{f}$ is continuous at $\mathbf{a}$ if and only if each f_i is continuous at $\mathbf{a}$ according to definition (12.14).
- $\mathbf{f}$ is continuous at $\mathbf{a}$ if and only if $\mathbf{f}(\mathbf{x}_k) \to \mathbf{f}(\mathbf{a})$ for every sequence $\{\mathbf{x}_k\}$ in M that converges to $\mathbf{a}$.

Characterizations of a continuous vector function of n variables.