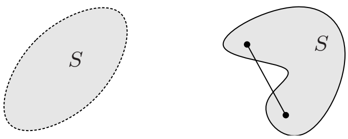
Chapter 13

Convexity

13.1 A set S in $\mathbb{R}^n$ is *convex* if

$$\mathbf{x}, \mathbf{y} \in S \text{ and } \lambda \in [0,1] \;\Rightarrow\; \lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in S$$

Definition of a convex set. The empty set is, by definition, convex.

13.2 

The first set is convex, while the second is not convex.

13.3 If S and T are convex sets in $\mathbb{R}^n$, then

- $S \cap T = \{\mathbf{x} : \mathbf{x} \in S \text{ and } \mathbf{x} \in T\}$ is convex
- $aS + bT = \{a\mathbf{s} + b\mathbf{t} : \mathbf{s} \in S,\ \mathbf{t} \in T\}$ is convex

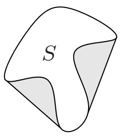
Properties of convex sets. (a and b are real numbers.)

13.4 Any vector $\mathbf{x} = \lambda_1\mathbf{x}_1 + \cdots + \lambda_m\mathbf{x}_m$, where $\lambda_i \geq 0$ for $i = 1, \ldots, m$ and $\sum_{i=1}^{m} \lambda_i = 1$, is called a *convex combination* of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m$ in $\mathbb{R}^n$.

Definition of a convex combination of vectors.

13.5 $$\operatorname{co}(S) = \begin{cases} \text{the set of all convex combinations of} \\ \text{finitely many vectors in } S. \end{cases}$$

$\operatorname{co}(S)$ is the *convex hull* of a set S in $\mathbb{R}^n$.

13.6 

If S is the unshaded set, then $\operatorname{co}(S)$ includes the shaded parts in addition.

13.7 $\operatorname{co}(S)$ is the smallest convex set containing S.

A useful characterization of the convex hull.

13.8 If $S \subset \mathbb{R}^n$ and $\mathbf{x} \in \operatorname{co}(S)$, then $\mathbf{x}$ is a convex combination of at most $n+1$ points in S.

Carathéodory's theorem.

13.9 $\mathbf{z}$ is an *extreme point* of a convex set S if $\mathbf{z} \in S$ and there are no $\mathbf{x}$ and $\mathbf{y}$ in S and λ in $(0,1)$ such that $\mathbf{x} \neq \mathbf{y}$ and $\mathbf{z} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}$.

Definition of an extreme point.

13.10 Let S be a compact, convex set in $\mathbb{R}^n$. Then S is the convex hull of its extreme points.

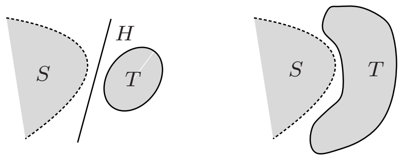
Krein–Milman's theorem.

13.11 Let S and T be two disjoint non-empty convex sets in $\mathbb{R}^n$. Then S and T can be separated by a hyperplane, i.e. there exists a non-zero vector $\mathbf{a}$ such that

$$\mathbf{a} \cdot \mathbf{x} \leq \mathbf{a} \cdot \mathbf{y} \quad \text{for all } \mathbf{x} \text{ in } S \text{ and all } \mathbf{y} \text{ in } T$$

Minkowski's separation theorem. A hyperplane $\{\mathbf{x} : \mathbf{a} \cdot \mathbf{x} = A\}$, with $\mathbf{a} \cdot \mathbf{x} \leq A \leq \mathbf{a} \cdot \mathbf{y}$ for all $\mathbf{x}$ in S and all $\mathbf{y}$ in T, is called *separating*.

13.12



In the first figure S and T are (strictly) separated by H. In the second, S and T cannot be separated by a hyperplane.

13.13 Let S be a convex set in $\mathbb{R}^n$ with interior points and let T be a convex set in $\mathbb{R}^n$ such that no point in $S \cap T$ (if there are any) is an interior point of S. Then S and T can be separated by a hyperplane, i.e. there exists a vector $\mathbf{a} \neq \mathbf{0}$ such that

$$\mathbf{a} \cdot \mathbf{x} \leq \mathbf{a} \cdot \mathbf{y} \text{ for all } \mathbf{x} \text{ in } S \text{ and all } \mathbf{y} \text{ in } T.$$

A general separation theorem in $\mathbb{R}^n$.

Concave and convex functions

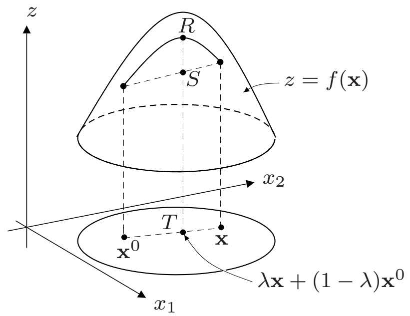
13.14 $f(\mathbf{x}) = f(x_1, \dots, x_n)$ defined on a convex set S in $\mathbb{R}^n$ is *concave* on S if

$$f(\lambda\mathbf{x} + (1-\lambda)\mathbf{x}^0) \geq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{x}^0)$$

for all $\mathbf{x}$, $\mathbf{x}^0$ in S and all λ in $(0,1)$.

To define a *convex* function, reverse the inequality. Equivalently, f is convex if and only if $-f$ is concave.

13.15



The function $f(\mathbf{x})$ is (strictly) concave. $TR = f(\lambda\mathbf{x}+(1-\lambda)\mathbf{x}^0) \geq TS = \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{x}^0)$. ($TR$ and TS are the heights of R and S above the $\mathbf{x}$-plane. The heights are negative if the points are below the $\mathbf{x}$-plane.)