Chapter 14

Classical optimization

14.1 $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ has a *maximum (minimum)* at $\mathbf{x}^* = (x_1^*, \ldots, x_n^*) \in S$ if

$$f(\mathbf{x}^*) - f(\mathbf{x}) \geq 0 \;\; (\leq 0) \quad \text{for all } \mathbf{x} \text{ in } S$$

$\mathbf{x}^*$ is called a *maximum (minimum) point* and $f(\mathbf{x}^*)$ is called a *maximum (minimum) value*.

Definition of (global) maximum (minimum) of a function of n variables. As collective names, we use *optimal* points and values, or *extreme* points and values.

14.2 $\mathbf{x}^*$ maximizes $f(\mathbf{x})$ over S if and only if $\mathbf{x}^*$ minimizes $-f(\mathbf{x})$ over S.

Used to convert minimization problems to maximization problems.

14.3

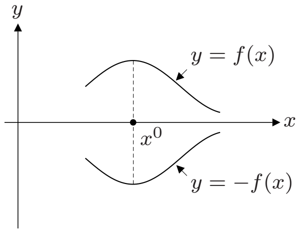


Illustration of (14.2). x^* maximizes $f(x)$ if and only if x^* minimizes $-f(x)$

14.4 Suppose $f(\mathbf{x})$ is defined on $S \subset \mathbb{R}^n$ and that $F(u)$ is strictly increasing on the range of f. Then $\mathbf{x}^*$ maximizes (minimizes) $f(\mathbf{x})$ on S if and only if $\mathbf{x}^*$ maximizes (minimizes) $F(f(\mathbf{x}))$ on S.

An important fact.

14.5 If $f : S \to \mathbb{R}$ is continuous on a closed, bounded set S in $\mathbb{R}^n$, then there exist maximum and minimum points for f in S.

The *extreme value theorem* (or *Weierstrass's theorem*).

14.6 $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ is a *stationary* point of $f(\mathbf{x})$ if

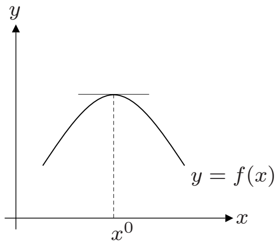
$$f_1'(\mathbf{x}^*) = 0, \;\; f_2'(\mathbf{x}^*) = 0, \;\; \ldots, \;\; f_n'(\mathbf{x}^*) = 0$$

Definition of stationary points for a differentiable function of n variables.

14.7 Let $f(\mathbf{x})$ be concave (convex) and defined on a convex set S in $\mathbb{R}^n$, and let $\mathbf{x}^*$ be an interior point of S. Then $\mathbf{x}^*$ maximizes (minimizes) $f(\mathbf{x})$ on S, if and only if $\mathbf{x}^*$ is a stationary point.

Maximum (minimum) of a concave (convex) function.

14.8



One-variable illustration of (14.7). f is concave, $f'(x^*) = 0$, and x^* is a maximum point.

14.9 If $f(\mathbf{x})$ has a maximum or minimum in $S \subset \mathbb{R}^n$, then the maximum/minimum points are found among the following points:

- interior points of S that are stationary
- extreme points of f at the boundary of S
- points in S where f is not differentiable

Where to find (global) maximum or minimum points.

14.10 $f(\mathbf{x})$ has a *local* maximum (minimum) at $\mathbf{x}^*$ if

$(*) \quad f(\mathbf{x}^*) - f(\mathbf{x}) \geq 0 \;\; (\leq 0)$

for all $\mathbf{x}$ in S sufficiently close to $\mathbf{x}^*$. More precisely, there exists an n-ball $B(\mathbf{x}^*; r)$ such that $(*)$ holds for all $\mathbf{x}$ in $S \cap B(\mathbf{x}^*; r)$.

Definition of local (or *relative*) maximum (minimum) points of a function of n variables. A collective name is *local extreme points*.

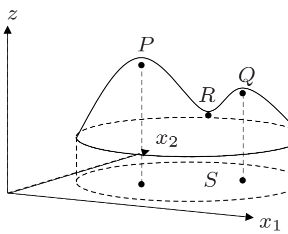
14.11 If $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ has a local maximum (minimum) at an interior point $\mathbf{x}^*$ of S, then $\mathbf{x}^*$ is a stationary point of f.

The *first-order conditions* for differentiable functions.

14.12 A stationary point $\mathbf{x}^*$ of $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ is called a *saddle point* if it is neither a local maximum point nor a local minimum point, i.e. if every n-ball $B(\mathbf{x}^*; r)$ contains points $\mathbf{x}$ such that $f(\mathbf{x}) < f(\mathbf{x}^*)$ and other points $\mathbf{z}$ such that $f(\mathbf{z}) > f(\mathbf{x}^*)$.

Definition of a saddle point.

14.13



The points P, Q, and R are all stationary points. P is a maximum point, Q is a local maximum point, whereas R is a saddle point.