Chapter 15

Linear and nonlinear programming

Linear programming

15.1
$$\max z = c_1x_1 + \cdots + c_nx_n \text{ subject to}$$
$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &\le b_2 \\ \cdots\cdots\cdots&\cdots\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &\le b_m \\ x_1 \ge 0, \ldots, x_n &\ge 0 \end{aligned}$$

A *linear programming problem*. (The *primal* problem.) $\sum_{j=1}^{n} c_jx_j$ is called the *objective function*. $(x_1, \ldots, x_n)$ is *admissible* if it satisfies all the $m+n$ constraints.

15.2
$$\min Z = b_1\lambda_1 + \cdots + b_m\lambda_m \text{ subject to}$$
$$\begin{aligned} a_{11}\lambda_1 + \cdots + a_{m1}\lambda_m &\ge c_1 \\ a_{12}\lambda_1 + \cdots + a_{m2}\lambda_m &\ge c_2 \\ \cdots\cdots\cdots&\cdots\cdots\cdots \\ a_{1n}\lambda_1 + \cdots + a_{mn}\lambda_m &\ge c_n \\ \lambda_1 \ge 0, \ldots, \lambda_m &\ge 0 \end{aligned}$$

The *dual* of (15.1). $\sum_{i=1}^{m} b_i\lambda_i$ is called the *objective function*. $(\lambda_1, \ldots, \lambda_m)$ is *admissible* if it satisfies all the $n + m$ constraints.

15.3
$$\max\ \mathbf{c}'\mathbf{x} \text{ subject to } \mathbf{A}\mathbf{x} \le \mathbf{b},\ \mathbf{x} \ge \mathbf{0}$$
$$\min\ \mathbf{b}'\boldsymbol{\lambda} \text{ subject to } \mathbf{A}'\boldsymbol{\lambda} \ge \mathbf{c},\ \boldsymbol{\lambda} \ge \mathbf{0}$$

Matrix formulations of (15.1) and (15.2). $\mathbf{A} = (a_{ij})_{m\times n}$, $\mathbf{x} = (x_j)_{n\times 1}$, $\boldsymbol{\lambda} = (\lambda_i)_{m\times 1}$, $\mathbf{c} = (c_j)_{n\times 1}$, $\mathbf{b} = (b_i)_{m\times 1}$.

15.4 If $(x_1, \ldots, x_n)$ and $(\lambda_1, \ldots, \lambda_m)$ are admissible in (15.1) and (15.2), respectively, then
$$b_1\lambda_1 + \cdots + b_m\lambda_m \ge c_1x_1 + \cdots + c_nx_n$$

The value of the objective function in the dual is always greater than or equal to the value of the objective function in the primal.

15.5 Suppose $(x_1^*, \dots, x_n^*)$ and $(\lambda_1^*, \dots, \lambda_m^*)$ are admissible in (15.1) and (15.2) respectively, and that

$$c_1 x_1^* + \cdots + c_n x_n^* = b_1 \lambda_1^* + \cdots + b_m \lambda_m^*$$

Then $(x_1^*, \dots, x_n^*)$ and $(\lambda_1^*, \dots, \lambda_m^*)$ are optimal in the respective problems.

An interesting result.

15.6 If either of the problems (15.1) and (15.2) has a finite optimal solution, so has the other, and the corresponding values of the objective functions are equal. If either problem has an "unbounded optimum", then the other problem has no admissible solutions.

The *duality theorem* of linear programming.

15.7 Consider problem (15.1). If we change b_i to $b_i + \Delta b_i$ for $i = 1, \dots, m$, and if the associated dual problem still has the same optimal solution, $(\lambda_1^*, \dots, \lambda_m^*)$, then the change in the optimal value of the objective function of the primal problem is

$$\Delta z^* = \lambda_1^* \Delta b_1 + \cdots + \lambda_m^* \Delta b_m$$

An important sensitivity result. (The dual problem usually *will* have the same solution if $|\Delta b_1|, \dots, |\Delta b_m|$ are sufficiently small.)

15.8 The ith optimal dual variable λ_i^* is equal to the change in objective function of the primal problem (15.1) when b_i is increased by one unit.

Interpretation of λ_i^* as a *"shadow price"*. (A special case of (15.7), with the same qualifications.)

15.9 Suppose that the primal problem (15.1) has an optimal solution $(x_1^*, \dots, x_n^*)$ and that the dual (15.2) has an optimal solution $(\lambda_1^*, \dots, \lambda_m^*)$. Then for $i = 1, \dots, n$, $j = 1, \dots, m$:

(1) $x_j^* > 0 \Rightarrow a_{1j}\lambda_1^* + \cdots + a_{mj}\lambda_m^* = c_j$

(2) $\lambda_i^* > 0 \Rightarrow a_{i1}x_1^* + \cdots + a_{in}x_n^* = b_i$

Complementary slackness. ((1): If the optimal variable j in the primal is positive, then restriction j in the dual is an equality at the optimum. (2) has a similar interpretation.)

15.10 Let $\mathbf{A}$ be an $m \times n$-matrix and $\mathbf{b}$ an n-vector. Then there exists a vector $\mathbf{y}$ with $\mathbf{Ay} \geq \mathbf{0}$ and $\mathbf{b}'\mathbf{y} < 0$ if and only if there is no $\mathbf{x} \geq \mathbf{0}$ such that $\mathbf{A}'\mathbf{x} = \mathbf{b}$.

Farkas's lemma.

Nonlinear programming

15.11 $\max \ f(x, y) \ \text{ subject to } \ g(x, y) \leq b$

A *nonlinear programming problem.*