# Chapter 16

# Calculus of variations and optimal control theory

## Calculus of variations

16.1 The simplest problem in the calculus of variations ($t_0$, $t_1$, $x^0$, and $x^1$ are fixed numbers):

$$\max \int_{t_0}^{t_1} F(t, x, \dot{x})\, dt, \quad x(t_0) = x^0, \quad x(t_1) = x^1$$

$F$ is a $C^2$ function. The unknown $x = x(t)$ is *admissible* if it is $C^1$ and satisfies the two boundary conditions. To handle the minimization problem, replace $F$ by $-F$.

16.2 $$\frac{\partial F}{\partial x} - \frac{d}{dt}\Big(\frac{\partial F}{\partial \dot{x}}\Big) = 0$$

The *Euler equation.* A necessary condition for the solution of (16.1).

16.3 $$\frac{\partial^2 F}{\partial \dot{x} \partial \dot{x}} \cdot \ddot{x} + \frac{\partial^2 F}{\partial x \partial \dot{x}} \cdot \dot{x} + \frac{\partial^2 F}{\partial t \partial \dot{x}} - \frac{\partial F}{\partial x} = 0$$

An alternative form of the Euler equation.

16.4 $F''_{\dot{x}\dot{x}}(t, x(t), \dot{x}(t)) \le 0$ for all $t$ in $[t_0, t_1]$

The *Legendre condition.* A necessary condition for the solution of (16.1).

16.5 If $F(t, x, \dot{x})$ is concave in $(x, \dot{x})$, an admissible function $x = x(t)$ that satisfies the Euler equation, solves problem (16.1).

Sufficient conditions for the solution of (16.1).

16.6 $x(t_1)$ free in (16.1) $\Rightarrow$ $\left[\dfrac{\partial F}{\partial \dot{x}}\right]_{t=t_1} = 0$

*Transversality condition.* Adding condition (16.5) gives sufficient conditions.

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| 16.7 | $x(t_1) \geq x^1$ in (16.1) $\Rightarrow$ $\left[\dfrac{\partial F}{\partial \dot{x}}\right]_{t=t_1} \leq 0 \quad (= 0 \text{ if } x(t_1) > x^1)$ | Transversality condition. Adding condition (16.5) gives sufficient conditions. |
| 16.8 | $t_1$ free in (16.1) $\Rightarrow$ $\left[F - \dot{x}\dfrac{\partial F}{\partial \dot{x}}\right]_{t=t_1} = 0$ | Transversality condition. |
| 16.9 | $x(t_1) = g(t_1)$ in (16.1) $\Rightarrow$ $\left[F + (\dot{g} - \dot{x})\dfrac{\partial F}{\partial \dot{x}}\right]_{t=t_1} = 0$ | Transversality condition. $g$ is a given $C^1$ function. |
| 16.10 | $\max\Big[\displaystyle\int_{t_0}^{t_1} F(t, x, \dot{x})\, dt + S(x(t_1))\Big], \quad x(t_0) = x^0$ | A variational problem with a $C^1$ *scrap value function*, $S$. |
| 16.11 | $\left[\dfrac{\partial F}{\partial \dot{x}}\right]_{t=t_1} + S'(x(t_1)) = 0$ | A solution to (16.10) must satisfy (16.2) and this transversality condition. |
| 16.12 | If $F(t, x, \dot{x})$ is concave in $(x, \dot{x})$ and $S(x)$ is concave, then an admissible function satisfying the Euler equation and (16.11) solves problem (16.10). | Sufficient conditions for the solution to (16.10). |
| 16.13 | $\max \displaystyle\int_{t_0}^{t_1} F\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \ldots, \frac{d^nx}{dt^n}\right) dt$ | A variational problem with higher order derivatives. (Boundary conditions are unspecified.) |
| 16.14 | $\dfrac{\partial F}{\partial x} - \dfrac{d}{dt}\left(\dfrac{\partial F}{\partial \dot{x}}\right) + \cdots + (-1)^n \dfrac{d^n}{dt^n}\left(\dfrac{\partial F}{\partial x^{(n)}}\right) = 0$ | The *(generalized) Euler equation* for (16.13). |
| 16.15 | $\max \displaystyle\iint_R F\left(t, s, x, \frac{\partial x}{\partial t}, \frac{\partial x}{\partial s}\right) dt\, ds$ | A variational problem in which the unknown $x(t, s)$ is a function of two variables. (Boundary conditions are unspecified.) |
| 16.16 | $\dfrac{\partial F}{\partial x} - \dfrac{\partial}{\partial t}\left(\dfrac{\partial F}{\partial x'_t}\right) - \dfrac{\partial}{\partial s}\left(\dfrac{\partial F}{\partial x'_s}\right) = 0$ | The (generalized) Euler equation for (16.15). |