Chapter 17

Discrete dynamic optimization

Dynamic programming

17.1 $$\max \sum_{t=0}^{T} f(t, \mathbf{x}_t, \mathbf{u}_t)$$
$$\mathbf{x}_{t+1} = \mathbf{g}(t, \mathbf{x}_t, \mathbf{u}_t), \quad t = 0, \dots, T-1$$
$$\mathbf{x}_0 = \mathbf{x}^0, \ \mathbf{x}_t \in \mathbb{R}^n, \ \mathbf{u}_t \in U \subset \mathbb{R}^r, \quad t = 0, \dots, T$$

A *dynamic programming problem*. Here $\mathbf{g} = (g_1, \dots, g_n)$, and $\mathbf{x}^0$ is a fixed vector in $\mathbb{R}^n$. U is the *control region*.

17.2 $$J_s(\mathbf{x}) = \max_{\mathbf{u}_s, \dots, \mathbf{u}_T \in U} \sum_{t=s}^{T} f(t, \mathbf{x}_t, \mathbf{u}_t), \text{ where}$$
$$\mathbf{x}_{t+1} = \mathbf{g}(t, \mathbf{x}_t, \mathbf{u}_t), \quad t = s, \dots, T-1, \quad \mathbf{x}_s = \mathbf{x}$$

Definition of the *value function*, $J_s(\mathbf{x})$, of problem (17.1).

17.3 $$J_T(\mathbf{x}) = \max_{\mathbf{u} \in U} f(T, \mathbf{x}, \mathbf{u})$$
$$J_s(\mathbf{x}) = \max_{\mathbf{u} \in U} \big[f(s, \mathbf{x}, \mathbf{u}) + J_{s+1}(\mathbf{g}(s, \mathbf{x}, \mathbf{u})) \big]$$
for $s = 0, 1, \dots, T-1$.

The *fundamental equations* in dynamic programming. (Bellman's equations.)

17.4 A "control parameter free" formulation of the dynamic programming problem:
$$\max \sum_{t=0}^{T} F(t, \mathbf{x}_t, \mathbf{x}_{t+1})$$
$$\mathbf{x}_{t+1} \in \Gamma_t(\mathbf{x}_t), \quad t = 0, \dots, T, \quad \mathbf{x}_0 \text{ given}$$

The set $\Gamma_t(\mathbf{x}_t)$ is often defined in terms of vector inequalities, $\mathbf{G}(t, \mathbf{x}_t) \le \mathbf{x}_{t+1} \le \mathbf{H}(t, \mathbf{x}_t)$, for given vector functions $\mathbf{G}$ and $\mathbf{H}$.

17.5 $J_s(\mathbf{x}) = \max \sum_{t=s}^{T} F(t, \mathbf{x}_t, \mathbf{x}_{t+1})$, where the maximum is taken over all $\mathbf{x}_{t+1}$ in $\Gamma_t(\mathbf{x}_t)$ for $t = s, \dots, T$, with $\mathbf{x}_s = \mathbf{x}$.

The *value function*, $J_s(\mathbf{x})$, of problem (17.4).

17.6 $$J_T(\mathbf{x}) = \max_{\mathbf{y} \in \Gamma_T(\mathbf{x})} F(T, \mathbf{x}, \mathbf{y})$$
$$J_s(\mathbf{x}) = \max_{\mathbf{y} \in \Gamma_s(\mathbf{x})} \big[F(s, \mathbf{x}, \mathbf{y}) + J_{s+1}(\mathbf{y}) \big]$$
for $s = 0, 1, \dots, T$.

The *fundamental equations* for problem (17.4).

17.7 If $\{\mathbf{x}_0^*, \ldots, \mathbf{x}_{T+1}^*\}$ is an optimal solution of problem (17.4) in which $\mathbf{x}_{t+1}^*$ is an interior point of $\Gamma_t(\mathbf{x}_t^*)$ for all t, and if the correspondence $\mathbf{x} \mapsto \complement\Gamma_t(\mathbf{x})$ is upper hemicontinuous, then $\{\mathbf{x}_0^*, \ldots, \mathbf{x}_{T+1}^*\}$ satisfies the *Euler vector difference equation*

$$F_{\mathbf{2}}'(t+1, \mathbf{x}_{t+1}, \mathbf{x}_{t+2}) + F_{\mathbf{3}}'(t, \mathbf{x}_t, \mathbf{x}_{t+1}) = 0$$

F is a function of $1+n+n$ variables, $F_{\mathbf{2}}'$ denotes the n-vector of partial derivatives of F w.r.t. variables no. 2, 3, ..., $n+1$, and $F_{\mathbf{3}}'$ is the n-vector of partial derivatives of F w.r.t. variables no. $n+2$, $n+3$, ..., $2n+1$.

Infinite horizon

17.8

$$\max \sum_{t=0}^{\infty} \alpha^t f(\mathbf{x}_t, \mathbf{u}_t)$$

$$\mathbf{x}_{t+1} = \mathbf{g}(\mathbf{x}_t, \mathbf{u}_t), \quad t = 0, 1, 2, \ldots$$

$$\mathbf{x}_0 = \mathbf{x}^0, \ \mathbf{x}_t \in \mathbb{R}^n, \ \mathbf{u}_t \in U \subset \mathbb{R}^r, \quad t = 0, 1, 2, \ldots$$

An infinite horizon problem. $\alpha \in (0,1)$ is a constant discount factor.

17.9 The sequence $\{(\mathbf{x}_t, \mathbf{u}_t)\}$ is called *admissible* if $\mathbf{u}_t \in U$, $\mathbf{x}_0 = \mathbf{x}^0$, and the difference equation in (17.8) is satisfied for all $t = 0, 1, 2, \ldots$.

Definition of an *admissible* sequence.

17.10

(B) $M \le f(\mathbf{x}, \mathbf{u}) \le N$

(BB) $f(\mathbf{x}, \mathbf{u}) \ge M$

(BA) $f(\mathbf{x}, \mathbf{u}) \le N$

Boundedness conditions. M and N are given numbers.

17.11 $V(\mathbf{x}, \boldsymbol{\pi}, s, \infty) = \sum_{t=s}^{\infty} \alpha^t f(\mathbf{x}_t, \mathbf{u}_t)$, where $\boldsymbol{\pi} = (\mathbf{u}_s, \mathbf{u}_{s+1}, \ldots)$, with $\mathbf{u}_{s+k} \in U$ for $k = 0, 1, \ldots$, and with $\mathbf{x}_{t+1} = \mathbf{g}(\mathbf{x}_t, \mathbf{u}_t)$ for $t = s, s+1, \ldots$, and with $\mathbf{x}_s = \mathbf{x}$.

The *total utility* obtained from period s and onwards, given that the state vector is $\mathbf{x}$ at $t = s$.

17.12 $J_s(\mathbf{x}) = \sup_{\boldsymbol{\pi}} V(\mathbf{x}, \boldsymbol{\pi}, s, \infty)$ where the supremum is taken over all vectors $\boldsymbol{\pi} = (\mathbf{u}_s, \mathbf{u}_{s+1}, \ldots)$ with $\mathbf{u}_{s+k} \in U$, with $(\mathbf{x}_t, \mathbf{u}_t)$ admissible for $t \ge s$, and with $\mathbf{x}_s = \mathbf{x}$.

The *value function* of problem (17.8).

17.13

$$J_s(\mathbf{x}) = \alpha^s J_0(\mathbf{x}), \quad s = 1, 2, \ldots$$

$$J_0(\mathbf{x}) = \sup_{\mathbf{u} \in U} \{f(\mathbf{x}, \mathbf{u}) + \alpha J_0(\mathbf{g}(\mathbf{x}, \mathbf{u}))\}$$

Properties of the value function, assuming that at least one of the boundedness conditions in (17.10) is satisfied.