Chapter 18

Vectors in $\mathbb{R}^n$. Abstract spaces

18.1	$\mathbf{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix}, \dots, \mathbf{a}_m = \begin{pmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{pmatrix}$	m (column) vectors in $\mathbb{R}^n$.
18.2	If $x_1, x_2, \dots, x_m$ are real numbers, then $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_m\mathbf{a}_m$ is a *linear combination* of $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$.	Definition of a linear combination of vectors.
18.3	The vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$ in $\mathbb{R}^n$ are • *linearly dependent* if there exist numbers c_1, $c_2, \dots, c_m$, not all zero, such that $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_m\mathbf{a}_m = \mathbf{0}$ • *linearly independent* if they are not linearly dependent.	Definition of linear dependence and independence.
18.4	The vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$ in (18.1) are linearly independent if and only if the matrix $(a_{ij})_{n\times m}$ has rank m.	A characterization of linear independence for m vectors in $\mathbb{R}^n$. (See (19.23) for the definition of rank.)
18.5	The vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ in $\mathbb{R}^n$ are linearly independent if and only if $\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \neq 0$	A characterization of linear independence for n vectors in $\mathbb{R}^n$. (A special case of (18.4).)
18.6	A non-empty subset V of vectors in $\mathbb{R}^n$ is a *subspace* of $\mathbb{R}^n$ if $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 \in V$ for all $\mathbf{a}_1, \mathbf{a}_2$ in V and all numbers c_1, c_2.	Definition of a subspace.

18.7 If V is a subset of $\mathbb{R}^n$, then $\mathcal{S}[V]$ is the set of all linear combinations of vectors from V.

$\mathcal{S}[V]$ is called the *span* of V.

18.8 A collection of vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$ in a subspace V of $\mathbb{R}^n$ is a *basis* for V if the following two conditions are satisfied:

- $\mathbf{a}_1, \dots, \mathbf{a}_m$ are linearly independent
- $\mathcal{S}[\mathbf{a}_1, \dots, \mathbf{a}_m] = V$

Definition of a basis for a subspace.

18.9 The *dimension* $\dim V$, of a subspace V of $\mathbb{R}^n$ is the number of vectors in a basis for V. (Two bases for V always have the same number of vectors.)

Definition of the dimension of a subspace. In particular, $\dim \mathbb{R}^n = n$.

18.10 Let V be an m-dimensional subspace of $\mathbb{R}^n$.

- Any collection of m linearly independent vectors in V is a basis for V.
- Any collection of m vectors in V that spans V is a basis for V.

Important facts about subspaces.

18.11 The *inner product* of $\mathbf{a} = (a_1, \dots, a_m)$ and $\mathbf{b} = (b_1, \dots, b_m)$ is the number

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + \cdots + a_m b_m = \sum_{j=1}^{m} a_i b_i$$

Definition of the inner product, also called *scalar product* or *dot product*.

18.12

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

$$(\alpha \mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\alpha \mathbf{b}) = \alpha (\mathbf{a} \cdot \mathbf{b})$$

$$\mathbf{a} \cdot \mathbf{a} > 0 \iff \mathbf{a} \neq \mathbf{0}$$

Properties of the inner product. α is a scalar (i.e. a real number).

18.13

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

Definition of the *(Euclidean) norm* (or *length*) of a vector.

18.14

(a) $\|\mathbf{a}\| > 0$ for $\mathbf{a} \neq \mathbf{0}$ and $\|\mathbf{0}\| = 0$

(b) $\|\alpha \mathbf{a}\| = |\alpha| \, \|\mathbf{a}\|$

(c) $\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$

(d) $|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\| \cdot \|\mathbf{b}\|$

Properties of the norm. $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$, α is a scalar. (d) is the *Cauchy–Schwarz inequality*. $\|\mathbf{a} - \mathbf{b}\|$ is the *distance* between $\mathbf{a}$ and $\mathbf{b}$.

18.15 The *angle* φ between two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ is defined by

$$\cos \varphi = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|}, \qquad 0 \leq \varphi \leq \pi$$

Definition of the angle between two vectors in $\mathbb{R}^n$. The vectors $\mathbf{a}$ and $\mathbf{b}$ are called *orthogonal* if $\mathbf{a} \cdot \mathbf{b} = 0$.